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Quantum Spectra and Transition from Regular to Chaotic Classical Motion

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We compute numerically the eigenvalues of a family of two-dimensional Hamiltonians which give rise to regular or chaotic classical motion depending on the particular choice of parameters. A close relationship is found between the spectral statistics and the fraction of classical phase space covered by chaotic trajectories. In the extreme regular and chaotic cases the system displays Poisson and Gaussian-orthogonal-ensemble statistics, respectively. A one-parameter random-matrix model is proposed to describe the spectral statistics in intermediate situations.

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It is by now well understood that most nonintegrable Hamiltonian systems have regions in phase space where the classical motion is chaotic beside other regions where the motion is regular.[1] This fact has prompted much activity aimed at understanding the quantum mechanical consequences of chaotic motion and, in particular, the statistical properties of the eigenvalues of the Hamiltonian.[2] Probably for technical reasons, most of this research has been confined to the semiclassical limit. A few specific systems, however, have been studied in detail quantum mechanically. By extending the work of Berry,[3] Bohigas, Giannoni, and Schmit have provided exciting information on the level statistics of Sinai's billiard.[4] They showed that the spectrum of Sinai's billiard possesses correlation properties that are in agreement with those of the Gaussian orthogonal ensemble of random matrices (GOE). This has led to the suggestion that, under conditions yet to be specified, the statistics of energy levels in finite chaotic systems is universal.

It appears unlikely that the problem of level statistics can be solved by analytic techniques at present. Hence, confirmation of the universality hypothesis must come from numerical diagonalization of a wide class of Hamiltonians. Sinai's billiard has the extreme property of being ergodic, while for a more general system we expect that only a certain fraction of phase space participates in the chaotic motion. In this Letter, we undertake to study the quantum mechanical spectra of simple Hamiltonians the classical motion of which undergoes a transition, as a function of some parameter of the system, from completely chaotic to regular behavior. We will show that many features of the spectral statistics can be described in terms of a random-matrix model. As a subsidiary result we confirm the universality hypothesis of Bohigas, Giannoni, and Schmit[4] by demonstrating the validity of GOE statistics for a strongly chaotic system that is not ergodic.

As our object of study we choose a family of two-dimensional systems which we like to visualize as two particles moving in one-dimensional potential wells and interacting through a local potential. This choice is formally expressed as

$$H=\tfrac{1}{2}(p_1^2+p_2^2)+\mathscr{V}_1(x_1)+\mathscr{V}_2(x_2)+\mathscr{V}_{12}(x_1-x_2). \qquad (1)$$

Given the numerical methods we use (see below), the problem of obtaining a large number of reliable eigenvalues restricts our freedom in choosing the functional form of $\mathscr{V}$. On the other hand, the choice of potential is constrained by the requirement that the system be sufficiently far from integrable in order to allow for chaotic motion in a wide energy range. A suitable choice was found to be

$$\mathscr{V}_i(x) = V_i(\alpha_i x^2 + \beta_i x^4 + \gamma_i x^6). \tag{2}$$

The numerical values for V_i, α_i, β_i, and γ_i were chosen in such a way as to satisfy the aforementioned requirements and are given in Table I. In accordance with the same requirements, the system was quantized with $\hbar^2 = 0.2$.

The eigenvalues of (1) were computed by expanding in a basis of harmonic-oscillator wave functions and diagonalizing the resulting truncated matrix. The stability of this procedure, details of which will be published elsewhere, was checked by independent methods. It is very important that, for the purpose of analyzing spectral correlations, the uncertainty in the numerical value of each eigenvalue must be small compared to the local mean spacing. This means that the requirements on numerical accuracy become more stringent with increasing excitation energy even if the level density is roughly constant as is the case for the present system.

The statistical quantities we study are the same as those considered in Ref. 4: (i) the distribution of nearest-neighbor spacings and (ii) the Δ_3 statistic[4] measuring the long-range correlations of the spectrum. Both quantities refer to an unfolded spectrum with a local level density fluctuating around unity. We unfolded the spectrum numerically by means of a least-squares fit. (The results were found to be insensitive to the fitting function used.) In Ref. 4 the spectral information derived from several Hamiltonians was superimposed to form *one* large ensemble in order to reduce the effect of statistical fluctuations. For the present system, which represents a many-parameter family of Hamiltonians with various degrees of stochasticity, it is *a priori* difficult to make an unbiased choice of ensemble and we therefore consider it necessary to analyze the results for each Hamiltonian separately. We can, however, improve our statistics by considering jointly the states with positive and negative parity. While carrying statistically independent information, states with different parity in a given energy region surely possess the same correlation properties. For the Hamiltonian defined in Eqs. (1) and (2) with $\hbar^2 = 0.2$, we were able to obtain about 400 reliable eigenvalues for each parity. As the first 25 to 40 levels probe mainly the harmonic part of the potentials, they are subject to a different level statistics and had to be discarded. This gave an ensemble of 720 levels to analyze.

We now turn to the discussion of our results and consider the chaotic case first. For $V_{12} = 100$ we find almost complete classical chaos in the energy region occupied by the first 400 levels. The qualification "almost" refers to the fact that a few small islands of stability were discovered and others may have gone undetected. Nevertheless, we estimate that more than 95% of phase space is covered by a single chaotic trajectory. Figure 1(a) displays the

TABLE I. Parameters of the potential defined in Eq. (2).

i	V_i	α_i	β_i	γ_i
1	100	1.56	−0.61	0.32
2	100	0.69	−0.12	0.03
12	V_{12}	−1.00	0.25	0.08

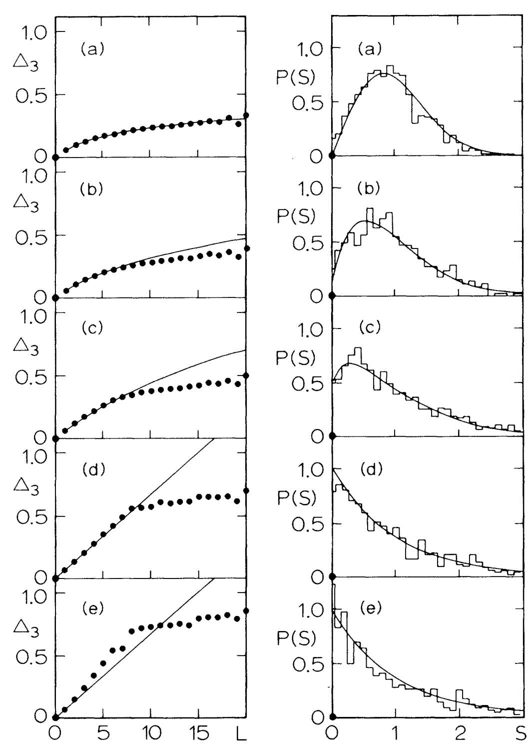

FIG. 1. Numerical results for the Δ_3 statistic and the distribution of nearest-neighbor spacings, $P(S)$. Dots and histograms represent the results obtained for the Hamiltonian (1) and lines those for a random-matrix model. (a) through (e) correspond to the order parameters ~ 1.0, ~ 0.8, ~ 0.6, ~ 0.1, and 0.0 in this order.

results for the distribution of nearest-neighbor spacings and the Δ_3 statistic averaged as in Ref. 4. We find no statistically significant deviation from GOE behavior [full line in Fig. 1(a)], thereby supporting and complementing the results of Ref. 4.

The present system becomes separable, and therefore integrable, on setting the interaction V_{12} equal to zero. It is generally believed that the eigenvalues of generic integrable systems obey Poisson statistics,[5] which implies a Poisson shape $[P(S)=e^{-S}]$ for the nearest-neighbor spacing distribution and a linear dependence of Δ_3, $\Delta_3(L) = L/15$. As is seen in Fig. 1(e), we find that the spacings are indeed distributed for $V_{12}=0$ according to e^{-S} although fluctuations around this distribution are noticeably larger than in the interacting (chaotic) case. The Δ_3 statistic follows the expected straight-line behavior up to $L \lesssim 9$. The slope is somewhat too large but this is due to a specific property of the present Hamiltonian. Other integrable members of the family yield different slopes close to $\frac{1}{15}$. However, for $L > 9$ the slope is systematically below the value required by Poisson statistics. Although further investigation is called for, we consider this discrepancy as an indication that the long-range correlations of the present integrable Hamiltonian are nongeneric. We have found numerically that the flattening of Δ_3 results from a particular form of level clustering which can be generated by folding two independent and weakly anharmonic spectra. Poisson statistics is recovered by making the anharmonicity sufficiently strong.

Situations intermediate between complete order and disorder can be attained by varying V_{12} between 0 and 100. In the long term, our aim is to establish a one-to-one correspondence between the spectral statistics and an appropriate classical "order parameter." We can imagine, at least, two quantities that may play the role of an order parameter: the fraction of phase space filled by chaotic trajectories and/or the Kolmogorov entropy. Given the simplicity of our system and the present amount of available data, the relevant order parameter describing the transition cannot be determined unequivocally. We therefore settle tentatively for the first of the two possibilities.

We now consider intermediate values of V_{12}, $V_{12}=40$, 30, and 10. The order parameters for these systems are 0.8 ± 0.1, 0.6 ± 0.1, and $\lesssim 0.1$. Results for Δ_3 and the nearest-neighbor spacing distributions are shown in Figs. 1(b)–1(d). We observe that for $V_{12}=10$ the system continues to yield essentially Poisson statistics (within the limitations discussed earlier) although the order parameter is already nonzero. The full lines in Figs. 1(b) and 1(c) are derived from a random-matrix model defined as a GOE with matrix elements M_{ij} modifed by a cutoff factor $\exp[-(i-j)^2/\sigma^2]$. As the effective bandwidth σ is varied, this model interpolates between Poisson $(\sigma=0)$ and GOE statistics $(\sigma \to \infty)$. Matrices of dimension 160 were used for this calculation. The bandwidths σ for $V_{12}=40$ and 30 were determined as $\sigma=7.5$ and 3.5 by fitting the Δ_3 statistic up to $L=6$, and we then used the same values for calculating the nearest-neighbor spacing distribution. Preliminary calculations indicate that the same qualitative behavior is obtained for other values of the parameters α_i, β_i, and γ_i. Further examples and a more complete discussion of the random-matrix model will be given in a later publication where we will also discuss the validity of an analytic theory of nearest-neighbor spacing distributions recently proposed by Berry and Robnik.[6]

In conclusion, by studying a family of simple Hamiltonians we found that there exists a close relationship between the degree of stochasticity of a classical system and the spectral statistics of the corresponding quantum system. We have proposed a one-parameter random-matrix model that represents the numerical data quite well although some discrepancies, which we ascribe to the nongeneric nature of our system, remain to be completely understood. Since the detailed form of the cutoff should have little influence on the level statistics, we expect that the present model may describe a wide range of physical systems.

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[1]A. J. Lichtenberg and M. A. Liebermann, *Regular and Stochastic Motion* (Springer, Berlin, 1983).

[2]G. M. Zaslavsky, Phys. Rep. **80**, 157 (1981).

[3]M. V. Berry, Ann. Phys. (N.Y.) **131**, 163 (1981).

[4]O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).

[5]M. V. Berry and M. Tabor, Proc. Roy. Soc. London, Ser. A **356**, 375 (1977).

[6]M. V. Berry and M. Robnik, J. Phys. A (to be published).