Growth of Order in a System with Continuous Symmetry

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The growth of order in a system with continuous symmetry is studied in the time-dependent Ginzburg-Landau model (in the large-N limit), quenched below the coexistence curve. The dynamics are studied with and without conservation of the order parameter. The development of order in the longitudinal direction and the buildup of Nambu-Goldstone modes in the transverse directions are explicitly exhibited. Nontrivial scaling behavior and growth laws are obtained in the long-time regime.

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In this paper we discuss the growth kinetics of a time-dependent Ginzburg-Landau model in the large-N limit which has been subjected to a rapid temperature quench[1] from very high temperatures to a temperature below the coexistence curve. We discuss the cases of both a conserved and a nonconserved order parameter. Our results are the first example[2] of an essentially exactly solvable model which shows nontrivial scaling behavior and growth laws for the development of order. It is also the first treatment of any kind of the growth of Nambu-Goldstone modes in systems growing order while globally maintaining a continuous symmetry.

The model that we treated is well known in the study of critical dynamics and is referred to as models A and B by Hohenberg and Halperin.[3] We have a vector order parameter $\psi_i(\vec{x},t)$ with N components. The equilibrium behavior of the system is governed by the Landau-Ginzburg-Wilson free energy

$$F=\frac{1}{2}\int d^dx\left[r\vec{\psi}^2+(\nabla\vec{\psi})^2+\frac{u}{2N}(\vec{\psi}^2)^2-\vec{H}\cdot\vec{\psi}\right], \tag{1}$$

where $H_i=H\delta_{i,z}$ is an external field. The dynamics of the field ψ_i are governed by the Langevin equation

$$\partial\psi_i(\vec{x},t)/\partial t=-\Gamma\,\delta F/\delta\psi_i+\eta_i, \tag{2}$$

where Γ is a kinetic coefficient. In the case where the order parameter is nonconserved (NCOP) Γ is a constant, while for the conserved order parameter (COP) the Fourier transform of the operator $\Gamma(\vec{x})$ is Dq^2 with wave number $\vec{q}$. The noise in (2), as usual, is Gaussianly distributed such that $\langle\eta_i\rangle=0$ and

$$\langle\eta_i(\vec{x},t)\eta_j(\vec{x}',t')\rangle = 2\Gamma(\vec{x})\delta(\vec{x}-\vec{x}')\delta(t-t')\delta_{ij}. \tag{3}$$

We are interested in the case where the system is initially in an equilibrium state characterized by the "temperature"[4] r_I and the field H_I. At some time $t_0=0$ one rapidly changes r and H to a set of final values r_F and H_F. The main quantities of interest are the average magnetization

$$m_i(t)=\langle\psi_i(\vec{x},t)\rangle, \tag{4}$$

and the structure factor $C(\vec{q},t)$ which is the Fourier transform of

$$C_i(\vec{x}-\vec{x}',t)=\langle\delta\psi_i(\vec{x},t)\delta\psi_i(\vec{x}',t)\rangle, \tag{5}$$

where $\delta\psi_i(\vec{x},t)=\psi_i(\vec{x},t)-m_i(t)$. The graphical structure associated with this model is well known[2,5] and it follows along standard lines that in the large-N limit we have the equations[6]

$$\partial M_i(t)/\partial t=-\Gamma(\vec{0})\{r_F M_i(t)-h_i^F+uM_i(t)[M^2(t)+S(t)]\}, \tag{6}$$

where $M_i(t)=m_i(t)/\sqrt{N}$, $h_i=H_i/\sqrt{N}$,

$$\zeta_i(t)=-r_F-u[M^2(t)+S(t)]-2u\,\delta_{i,z}M^2(t), \tag{7}$$

$$S(t)=\int d^dq\,(2\pi)^{-d}C_\perp(\vec{q},t), \tag{8}$$

and

$$\partial C_i(\vec{q},t)/\partial t=-2\Gamma(q)\{q^2-\zeta_i(t)\}C_i(\vec{q},t)+2\Gamma(q). \tag{9}$$

In the presence of the external field, or if the system is initially ordered, i labels the longitudinal direction, $i=z$, and the $(N-1)$ transverse directions, $i=\perp$. We consider the case where r_I is very large so that

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$C_i(\vec{q},0)=S(0)=0$. We have defined ζ_i such that with r_F large and negative, ζ_i is large and positive initially. For sufficiently small q^2 we see that the system will be initially unstable since ζ_i is positive. The system will respond to this instability by growing into the new equilibrium state.

While one can, of course, study the most general case $(H_I,r_I)\rightarrow(H_F,r_F)$ we have chosen to focus in this paper on two particular situations which we believe offer the most interesting results. The first case is a quench in zero field of a NCOP. In this case $H_I=H_F=0$ and one sees that $M(t)=0$, and there is no symmetry breaking. This situation is interesting because the system will still build order over arbitrarily large length scales and, as we shall see, develop Nambu-Goldstone modes. In the second case we consider a COP where we have a finite magnetization $M_i=M\delta_{i,z}$ initially. Consider dropping the temperature and adjusting H_F such that the average magnetization is fixed at the value M (analogous to holding the number of particles fixed in a lattice-gas model). As one approaches the coexistence curve H_F goes to zero. We look at the case where r_F takes one below the coexistence curve and $H_F=0$.

Case I.—In this case $M=0$, $\Gamma(q)=\Gamma$, and there is no distinction between longitudinal and transverse modes, $C_i=C(\vec{q},t)$. We must solve Eqs. (7)–(9). We have done this numerically[7] for $d=3$ and the primary result is physically transparent. The point is that for small q, $C(q,t)$ will begin to grow. Therefore, S will increase and $\zeta=|r_F|-uS(t)$ will decrease, lowering the threshold of the unstable modes. We find that $S(t)$, after an initial fast transient, slowly approaches the saturation value $|r_F|/u$. In this regime $\zeta(t)$ approaches zero as t^{-1}. Therefore, the system minimally restores its stability as $t\rightarrow\infty$. We then find, for times $t>10$, that the structure factor is given very accurately by

$$C(\vec{q},t)=\{1-\exp[-2\Gamma(\vec{q})q^2t]\}q^{-2}+AL^3(t)F(\vec{q}L(t)), \tag{10}$$

where, even though $m(t)=0$, the coefficient A rapidly evolves in time to the value, $A=M_F^2=|r_F|/u-S_c$, where M_F is the magnitude of the spontaneous magnetization associated with the final equilibrium state $(r_F,H_F=0)$ and $S_c=\int d^3q\,(2\pi)^{-3}q^{-2}$.

In order to explain the significance of Eq. (10) let us recall that the critical point of the model is given by $r_c=-uS_c$ and S_c gives an upper bound on the equilibrium order-parameter fluctuations in the absence of symmetry breaking. Since we are quenching the system well below the critical point ($|r_F|\gg|r_c|$) and $S(t)$ rapidly approaches the saturation value $|r_F|/u$, the system, after the initial fast transient, must simultaneously satisfy the conflicting needs of reaching stability and maintaining $S(t)\gg S_c$ without breaking the global symmetry. The way out is a gradual local symmetry breaking through the formation of domains, which generate a central peak in the structure factor and correspond to the second term on the right-hand side of Eq. (10). The inverse of the width of this peak is a length $L(t)$ associated with a typical domain size. We find that $L(t)\sim t^{1/2}$ for long times in agreement with the Lifshitz-Cahn-Allen curvature-driven growth law developed[8] for scalar order parameters. Thus the growth law appears to be independent of N, the number of components of the order parameter, in the NCOP case. We also find scaling behavior[9] of the type described in detail elsewhere. The shape function, shown in Fig. 1, is Gaussian $F(x)=\exp(-ax^2)$ for $x<4$ with $a=\ln 2$, and it is found to be independent of time and temperature. As $t\rightarrow\infty$ the scaling term in Eq. (10) goes over to a Bragg peak indicating symmetry breaking over domains of arbitrarily large size. Consequently, one can expect the appearance of Nambu-Goldstone modes $\sim q^{-2}$. This is described by the first term on the right-hand side of Eq. (10). The physical picture seems clear from an analysis of the $\vec{q}\simeq 0$ component of the structure factor, $C(\vec{0},t)=t+423.7t^{3/2}$, where the $t^{3/2}$ term corresponds to the setting up of domains of the new phases and the linear term corresponds to the

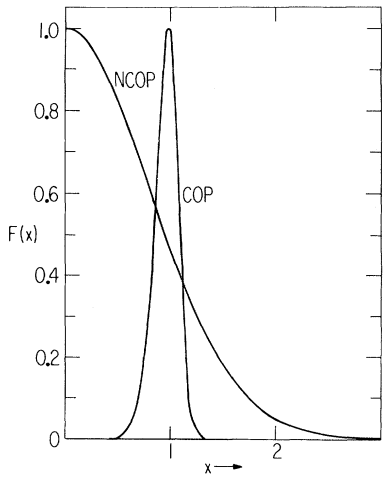

FIG. 1. Normalized shape functions defined in Eq. (10) for NCOP and COP.

development of Nambu-Goldstone modes within these domains.

Case II.—In this case $\Gamma(q) = Dq^2$, $\Gamma(0) = 0$, and $M_i(t) = M\delta_{i,z}$ is time independent and (6) is a trivial identity. In this case, for $M \neq 0$, we must solve (9) for $C_\perp(\vec{q},t)$ together with (7) for $\zeta_\perp = |r_F| - u \times [M_I^2 + S(t)]$ and (8). The longitudinal component $C_z(\vec{q},t)$ is then given by (9) once $\zeta_z(t) = \zeta_\perp(t) - 2uM^2$ is known. Again, the important result of the analysis is that $\zeta_\perp \sim t^{-1/2}$ for long times and we find that $C_\perp$ is given again by (10) but with several important differences. First, in the Nambu-Goldstone component, one has $2Dq^4t$ in the exponent and this piece has the characteristic peaked structure seen in spinodal decomposition. The peak height and position of this contribution are given by $P_{\rm NG} = 2.01t^{1/2}$ and $q_{\rm NG} = 0.6t^{-1/4}$. The peak position of the "Bragg peak" contribution is given by $q_m(t) = L^{-1} = 1.7t^{-1/4}$ for long times (in contrast with the Lifshitz-Slyozov result $t^{-1/3}$) and the peak height by $P_m(t) = 104.0t^{3/4}$, where the amplitudes of the power laws are for $r_F = -10$. In this case we find that $A = |r_F|/u - M^2 - S_c = M_F^2 - M^2$ and the shape function is shown in Fig. 1. For $x > 2$ the shape function is well approximated by the form $(a + bx^4)^{-1}$, where $a = -1.3 \times 10^4$ and $b = 6.1 \times 10^3$ in agreement with the speculation of Furukawa[10] in the scalar-order-parameter case. The longitudinal component simply equilibrates to the final form

$$C_z(\vec{q}, \infty) = (q^2 + 2uM^2)^{-1}. \quad (11)$$

Thus only the transverse component develops a *new* Bragg peak. This makes physical sense. Because of the conservation law and the initial ordering the longitudinal magnetization does not change after the quench. Thus if one adds up the magnetization in the z direction one only finds domains ordered as in the initial state. On the other hand, the equation of state in the final state must be satisfied, $r_F + uS_c + uM_F^2 = 0$ with $M_F^2 > M^2$. This is achieved by developing ordering in the perpendicular directions, where domains in all the transverse directions will order, giving contributions to a Bragg peak just as in case I. If we look at the Bragg peak contributions to the total magnetization squared we have a contribution M^2 from C_z and $M_F^2 - M^2$ from $C_\perp$ and, as expected, the sum is M_F^2.

We point out that our formulation here is formally very similar to the work of Langer, Bar-on, and Miller[11] (LBM) where they solve equations very similar to (10) for the case of a scalar order parameter. Binder, Billotet, and Mirold[12] pointed out that the LBM model led to an unphysical q^{-2} behavior in the structure factor. We see here, where our model involves no approximations, that this q^{-2} reflects the growth of Nambu-Goldstone modes. It seems then that the LBM model is not appropriate for systems with a scalar order parameter.

Our results here are somewhat different from those found recently[13] for the COP kinetic Ising model. A logarithmic long-time growth law was obtained there with use of Monte Carlo and renormalization-group methods. This logarithmic behavior was a manifestation of a freezing behavior for quenches to zero temperature. We expect no such freezing in the model that we study here (which has very-low-energy gapless excitations) and conclude that the long-time growth laws may therefore be substantially different for the two systems.

The relevance of our work to the physical cases of $N = 2$ and 3 seems clear in the NCOP case where the growth law for $L(t)$ is the same as found in the Ising $N = 1$ case. We speculate, therefore, that $L(t) \sim t^{1/2}$ for all N. The COP case is less clear since even the growth law for $N = 1$ for the time-dependent Ginzburg-Landau model is not yet known. Our results cannot be extrapolated directly to two dimensions for the obvious reason that there is no phase transition at finite temperatures for $N > 2$.

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