Lorentz Anomalies

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(Received 29 March 1984)

Associated with the local Lorentz symmetry of gravitational interactions of fermions there are chiral anomalies in $(4n+2)$-dimensional space-times. We present chiral Lorentz anomalies in two and six dimensions. Existence of chiral Lorentz anomalies implies that quantum effects induce a breakdown of local Lorentz symmetry in $4n+2$ dimensions, in general, in theories containing chiral Weyl fermions.

PACS numbers: 11.15.Ex, 04.20.Me, 11.30.Cp

Associated with internal gauge symmetries, there are the well-known axial-vector anomalies.[1-3] Fermions interacting with gravitational field have two symmetry properties: (i) general covariance; (ii) local Lorentz invariance. Are there "axial" anomalies associated with these gravitational symmetries? Alvarez-Gaumé and Witten[4] have recently analyzed the question with respect to general coordinate transformations and shown the existence of anomalies in $4n+2$ dimensions. We have, on the other hand, looked at the anomaly question with respect to local Lorentz symmetry, which is the closest gravitational counterpart[5] of the usual non-Abelian gauge symmetries. Anomalies also exist, again in $4n+2$ dimensions. In this note, we report our results for the case of spin-$\frac{1}{2}$ fermions.

Let us consider a massless Dirac spinor in an even N-dimensional Riemannian background of a classical gravitational field. The action is of the form[6]

$$W=\int d^N x(\det e_a{}^\mu)\tfrac{1}{2}(\bar\psi i\gamma^a e_a{}^\mu D_\mu\psi-\bar\psi\bar D_\mu i\gamma^a e_a{}^\mu\psi), \tag{1}$$

where

$$D_\mu=\vec\partial_\mu-i\tfrac{1}{4}\sigma_{ab}\omega^{ab}{}_\mu,\qquad \bar D_\mu=\overleftarrow\partial_\mu+i\tfrac{1}{4}\sigma_{ab}\omega^{ab}{}_\mu,$$

and $\omega^{ab}{}_\mu$ is the Lorentz connection, a known function of $e_a{}^\mu$. It is invariant under local Lorentz transformations with $\omega^{ab}{}_\mu$ serving as the corresponding gauge connection, and under the general coordinate transformations. Associated with these invariances are two separate conservation laws. The conservation equation implied by local Lorentz invariance is[6,7]

$$S_{ab}{}^\mu{}_{;\mu}+(t_{ab}-t_{ba})=0, \tag{2}$$

where

$$S_{ab}{}^\mu=\tfrac{1}{4}e^{c\mu}\bar\psi(\sigma_{ab}\gamma_c+\gamma_c\sigma_{ab})\psi, \tag{3}$$

$$t_{ab}=e_b{}^\mu t_{a\mu}=e_b^\mu\tfrac{1}{2}(\bar\psi i\gamma_a D_\mu\psi-\bar\psi-\bar\psi\bar D_\mu i\gamma_a\psi). \tag{4}$$

The one that follows from general covariance is

$$T^\nu{}_{\mu;\nu}-\tfrac{1}{2}\omega^{ab}{}_\mu(T_{ab}-T_{ba})=0, \tag{5}$$

where

$$T^\nu{}_\mu=e^{a\nu}e^b{}_\mu T_{ab},\qquad T_{ab}=t_{ab}+\tfrac{1}{2}S_{ab}{}^\mu{}_{;\mu}. \tag{6}$$

We shall in this note concentrate on anomalies associated with the local Lorentz invariance. The chiral counterpart of the conservation equation (2) is

$$^{(5)}S_{ab}{}^\mu{}_{;\mu}+[^{(5)}t_{ab}-{}^{(5)}t_{ba}]=0, \tag{7}$$

where ${}^{(5)}S_{ab}{}^{\mu}$ and ${}^{(5)}t_{ab}$ are obtained by inserting the Hermitian matrix

$$\Gamma_5 \equiv i^{(N-2)/2}\gamma^0\gamma^1 \cdots \gamma^{N-1}, \tag{8}$$

in front of ψ in (3) and (4). Equations (2) and (7) are conservation laws at the classical level. Quantum effects may modify these equations, giving rise to "anomalies." These Lorentz anomalies can be calculated by using the same techniques previously developed for axial-vector anomalies. We use two different methods of calculation. One is based on the Schwinger-DeWitt proper-time method coupled with the generalized ζ-function regularization procedure,[8] and the other on the path-integral measured *à la* Fujikawa.[9] As expected, there is no modification of (2) in any dimension. However, (7) is modified in $4n+2$ dimensions. That there are no anomalies in $4n$ dimensions is a consequence of the charge-conjugation properties of the theory.

The chiral Lorentz anomaly can be conveniently expressed in terms of DeWitt's proper-time expansion coefficients $a_{N/2}(x)$. When the anomalous term due to quantum effects is included, the equation (7) is modified to become[10]

$${}^{(5)}S_{ab}{}^{\mu\nu}{}_{;\mu} + [{}^{(5)}t_{ab} - {}^{(5)}t_{ba}] = i^N(4\pi)^{-N/2}\,\mathrm{Tr}(\Gamma_5\sigma_{ab}a_{N/2}). \tag{9}$$

To calculate the trace in (9), we only need terms with $(N-2)/2$ or more powers of σ_{cd}'s in $a_{N/2}$. Even so, the algebra is quite tedious when N is large. For $N=2$, the result is relatively simple: $a_1(x) = \frac{1}{12}R$, where R is the scalar curvature, giving rise to $(\epsilon_{01} = -\epsilon_{10} = -1)$

$${}^{(5)}S_{ab}{}^{\mu}{}_{;\mu} + [{}^{(5)}t_{ab} - {}^{(5)}t_{ba}] = -i\epsilon_{ab}(24\pi)^{-1}R \quad (N=2). \tag{10}$$

For $N=4$, one has[10]

$$\mathrm{Tr}(\gamma_5\sigma_{ab}a_2) = -\tfrac{1}{6}(R_a{}^{c\mu\nu}\tilde{R}_{bc\mu\nu} - R_b{}^{c\mu\nu}\tilde{R}_{ac\mu\nu}), \tag{11}$$

where $\tilde{R}_{ab\mu\nu} = \frac{1}{2}\epsilon_{abcd}R^{cd}{}_{\mu\nu}$, with $\epsilon_{0123} = -1$. The right-hand side of (11) is identically zero, as expected from the general considerations mentioned above.

For $N=6$, the algebra starts to be quite involved. The result is the following

$${}^{(5)}S_{ab}{}^{\mu}{}_{;\mu} + [{}^{(5)}t_{ab} - {}^{(5)}t_{ba}] = {}^{(5)}N_{ab}, \tag{12}$$

where $(\epsilon^{012345} = -\epsilon_{012345} = 1)$

$$\begin{aligned}{}^{(5)}N_{ab} = i\tfrac{1}{8}(4\pi)^{-3}\epsilon_{abcdef}[&-\tfrac{1}{80}RR^{cd\mu\nu}R^{ef}{}_{\mu\nu} + \tfrac{1}{36}R^{\mu\nu}R^{cd}{}_{\lambda\mu}R^{ef\lambda}{}_{\nu} + \tfrac{1}{72}R^{\mu\nu\lambda\zeta}R^{cd}{}_{\mu\nu}R^{ef}{}_{\lambda\zeta}\\ &+\tfrac{1}{45}R^{\mu\lambda\nu\zeta}R^{cd}{}_{\mu\nu}R^{ef}{}_{\lambda\zeta} - \tfrac{1}{60}(R^{cd}{}_{\mu\nu;}{}^{\mu}R^{ef\lambda\nu}{}_{;\lambda} + 4R^{cd\mu\nu}{}_{;}{}^{\lambda}R^{ef}{}_{\mu\nu;\lambda})\\ &+\tfrac{1}{30}(R^{cd}{}_{\lambda\mu;}{}^{\lambda}{}_{\nu} + R^{cd}{}_{\lambda\mu;\nu}{}^{\lambda} - R^{cd}{}_{\mu\nu;\lambda}{}^{\lambda})R^{ef\mu\nu}].\end{aligned} \tag{12'}$$

We have so far considered theories with Dirac spinors, for which the axial-type quantities ${}^{(5)}S_{ab}{}^{\mu}$, ${}^{(5)}t_{ab}$, etc., do not really appear in the Lagrangian (1), and the anomalies are, though interesting, not particularly significant. However, for a chiral Weyl spinor, with a definite chirality, the implication is significant. Consider a massless left-handed Weyl spinor ψ_L interacting with gravitational field, with the action denoted by W_L. The responses of W_L with respect to local Lorentz transformation (LLT) and general coordinate transformation (GCT) are, respectively,

$$\delta_{(\mathrm{LLT})}W_L = \int d^Nx\,(\det e_a{}^{\mu})\tfrac{1}{2}\delta\epsilon^{ab}(x)[{}^{(L)}S_{ab;\mu} + {}^{(L)}t_{ab} - {}^{(L)}t_{ba}], \tag{13}$$

and

$$\delta_{\mathrm{GCT}}W_L = -\int d^Nx\,(\det e_a{}^{\mu})[{}^{(L)}T^{\nu}{}_{\mu;\nu} - \omega^{ab}{}_{\mu}{}^{(L)}T_{ab}]\delta x^{\mu}, \tag{14}$$

where the expressions ${}^{(L)}S_{ab}{}^{\mu}$, etc., are obtained from the corresponding expressions (3), (4), and (6) by replacing ψ with ψ_L. Gravitational chiral anomalies, like

$${}^{(L)}S_{ab}{}^{\mu}{}_{;\mu} + [{}^{(L)}t_{ab} - {}^{(L)}t_{ba}] = {}^{(L)}N_{ab}, \tag{15}$$

then imply *a breakdown of the corresponding gravitational symmetries* as a result of quantum corrections.

The precise form of the Lorentz anomalies ${}^{(L)}N_{ab}$ for spin-$\frac{1}{2}$ chiral fermions can be deduced from an extension of the considerations leading to (9). For $N=4n+2$, they are given by

$$
\begin{aligned}
{}^{(\mathrm{L})}N_{ab} &= i^N(4\pi)^{-N/2}\mathrm{Tr}\left[\frac{1-\Gamma_5}{2}\sigma_{ab}a_{N/2}\right] \\
&= \tfrac{1}{2}\,{}^{(5)}N_{ab}.
\end{aligned} \tag{16}
$$

For $N=2$ and 6, ${}^{(5)}N_{ab}$ is given by (10) and (12′), respectively.

We close with a few remarks:

(i) According to (6) and (15),

$$
{}^{(L)}T_{ab} - {}^{(\mathrm{L})}T_{ba} = {}^{(\mathrm{L})}N_{ab}. \tag{17}
$$

The quantum corrections therefore induce in $4n+2$ dimensions a nonvanishing antisymmetric part of the energy-momentum tensor $T_{\mu\nu}$, which is actually equal to the Lorentz anomaly. It is to be noted that it is the symmetric part[11] of $T_{\mu\nu}$ that is the source of gravitational field $g_{\mu\nu}$ in the Einstein equation.

(ii) In $4n+2$ dimensions, there are two kinds of gravitational chiral anomalies, associated with the two gravitational symmetries. The anomaly associated with the general coordinate transformation[4] is of the general form

$$
{}^{(\mathrm{L})}T^{\nu}{}_{\mu;\nu} - \omega^{ab}{}_{\mu}\,{}^{(\mathrm{L})}T_{ab} = {}^{(\mathrm{L})}Q_{\mu}, \tag{18}
$$

which, on account of (17), can be written as

$$
T^{\nu}{}_{\mu;\nu} - \tfrac{1}{2}\omega^{ab}{}_{\mu}\,{}^{(\mathrm{L})}N_{ab} = {}^{(\mathrm{L})}Q_{\mu}. \tag{19}
$$

The chiral Lorentz anomaly ${}^{(\mathrm{L})}N_{ab}$ thus contributes to the general-coordinate anomaly ${}^{(\mathrm{L})}Q_{\mu}$.

(iii) The presence of the two gravitational anomalies places severe constraints on chiral gravitational theories in $4n+2$ dimensions. Presumably, both kinds of anomalies should get cancelled. It remains to be seen how such restraints are put to use to select models.

(iv) The Lorentz anomalies, except for $N=2$, depend on derivatives of the curvature tensor, and, consequently, on the intricate dynamics of the gravitational field. The dependence on derivatives probably also makes it harder to uncover the geometrical meaning of the Lorentz anomalies. This is a new feature of the Lorentz anomaly that is different from the usual chiral anomalies.

Details and other related discussions will be presented elsewhere.

We thank Professor C. N. Yang for his interest and encouragement. This work is partially supported by the National Science Foundation under Contracts No. NSF-PHY-82-07041 and No. NSF-PHY-81-09110.

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