Bardeen-Moshe-Bander Fixed Point and the Ultraviolet Triviality of $(\vec{\Phi}^2)^3_3$

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The three-dimensional $(\vec{\Phi}^2)^3$ theory is studied at large N. A complete mapping of the phase diagram and a detailed analysis of the renormalization-group flows is given. The recently found uv fixed point of Bardeen, Moshe, and Bander is investigated and it is pointed out that its characteristics seem to be somewhat nongeneric and might disappear at finite N. Our analysis is restricted to infinite N and hence no definitive conclusions are yet possible.

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This work presents a study of scalar O(N)-symmetric quantum field theories in three Euclidean dimensions in the large-N limit. For finite N, with a given renormalization-group (RG) transformation we expect the following picture to hold[1]: In the space of all interactions there is a hierarchy of isolated fixed points (FP), each located on the boundary of the domain of attraction of its predecessor. The most stable FP is noncritical (TFP) and corresponds to a theory with an infinitely massive particle. The most stable critical FP has one unstable direction and governs the critical behavior of the O(N) Heisenberg mode (HFP). The Gaussian FP (GFP) has two unstable directions (and a marginally stable one) and controls the tricritical behavior of the O(N) model. Different kinds of continuum field theories are associated with the various trajectories connecting the FP's. The theory constructed on the trajectory leading from the HFP to the TFP has only a scale as an adjustable parameter. It is complicated at high energy but qualitatively simple at low energy.[2] The two-dimensional unstable surface emanating from the GFP gives rise to a two-parameter family of theories. The control over the uv regime (given by the GFP) allows the perturbative construction of these superrenormalizable $(\vec{\Phi}^2)^2_3$ models.

At infinite N the marginally stable direction at the GFP becomes absolutely marginal and a line of fixed points appears, corresponding to $N=\infty$ scale-invariant $(\vec{\Phi}^2)^3$ theories. Recently Bardeen, Moshe, and Bander (BMB) have claimed, on the basis of a variational calculation, that this line ends at a new, nontrivial uv attractive FP, at which scale invariance is spontaneously broken, with the appearance of a dilaton.[3] It was implied that this new FP survives to finite N so that a $(\vec{\Phi}^2)^3_3$ continuum theory with three adjustable parameters and a nonperturbative high-energy behavior exists at finite N. The BMB FP supersedes the uv FP suggested by Townsend,[4] Pisarski,[4] and Appelquist and Heinz[4] on the basis of large-N perturbation theory.

In this paper we intend to exhibit explicitly the RG flows at $N=\infty$ in the neighborhood of the line of GFP's. For this purpose we shall first compute the effective potential by Euclidean functional methods and identify the various FP's by picking an (assumed generic) three-dimensional surface of bare interactions. On this surface we shall map out completely the phase diagram. The bare action is

$$A_B[\vec{\Phi}]=\int[\tfrac{1}{2}(\partial_\mu\vec{\Phi})^2+V_0(\vec{\Phi}^2)]d^3x,$$
$$V_0(\vec{\Phi}^2)=\tfrac{1}{2}\mu_0^2\vec{\Phi}^2+\tfrac{1}{4}\lambda_0(\vec{\Phi}^2)^2+\tfrac{1}{6}\eta_0(\vec{\Phi}^2)^3. \tag{1}$$

The $1/N$ expansion is obtained when we represent the interaction term as an inverse Laplace transform,

$$\exp[-V_0(\vec{\Phi}^2)]=(4i\pi)^{-1}\int_{-i\infty}^{i\infty}d\tilde{M}^2\int_0^\infty d\tilde{X}\exp[-V_0(\tilde{X})+\tfrac{1}{2}\tilde{M}^2(\tilde{X}-\vec{\Phi}^2)], \tag{2}$$

and perform the Gaussian $\vec{\Phi}$ integration. The N dependence then becomes explicit, and, at leading order in $1/N$, one is left with a saddle-point problem in the composite fields $\tilde{M}^2$ and $\tilde{X}$. The effective potential $V_{\text{eff}}(\vec{\Phi}_c^2)$ is obtained when we add a space-independent source term and perform the standard Legendre transform. The variational method employed by BMB gives equivalent results at infinite N, but we feel that our approach is more standard and that its correctness at $N=\infty$ is more transparent.

We work with a sharp momentum cutoff Λ and rescale all the fields and the potential in order to work with dimensionless quantities:

$$\vec{\phi} = (\Lambda N)^{-1/2}\vec{\Phi}, \quad X = (\Lambda N)^{-1}\tilde{X},$$
$$M^2 = \Lambda^{-2}\tilde{M}^2, \quad V_0(\vec{\Phi}^2) = N\Lambda^3 U_0(\vec{\phi}^2). \tag{3}$$

Similar rescalings define $\vec{\phi}_c$ and $U_{\rm eff}(\vec{\phi}_c^2)$. Up to a constant, the $N=\infty$ effective potential is given by

$$U_{\rm eff}(\vec{\phi}_c^2) = -\frac{1}{2}\int_{\rm const}^{X} g(t)dt + U_0(\vec{\phi}_c^2 + X), \tag{4a}$$

$$M^2 = g(X) = 2U_0'(\vec{\phi}_c^2 + X), \tag{4b}$$

where the function g is defined in $]0, 1/2\pi^2]$ as the inverse of f:

$$X = f(M^2) = \int_{|q|<1} \frac{d^3q}{(2\pi)^3}\frac{1}{q^2+M^2} = (2\pi^2)^{-1}[1 - |M|\tan^{-1}(1/|M|)]. \tag{5}$$

If there are many solutions to (4b) the relevant one is the absolute minimum of (4a). The vacuum corresponds to the minimum of $U_{\rm eff}$ given by the equation $M^2\vec{\phi}_c = 0$ with two solutions: The O(N)-symmetric one has $\vec{\phi}_c = 0$ and $M^2 > 0$, while in the broken phase we get $\vec{\phi}_c \neq 0$ and $M^2 = 0$. For the vacuum we extend the definition of $g(x)$ to $x > 1/2\pi^2$, where it is taken to vanish. Extrema of the effective potential correspond now to solutions $\bar{X}$ of $g(X) = 2U_0'(X)$, and the corresponding value of $U_{\rm eff}$ is given by the integral of $U_0' - 1/2g$ from 0 to $\bar{X}$. If $\bar{X} < 1/2\pi^2$ we are in the symmetric phase and if $\bar{X} > 1/2\pi^2$ we are in the broken phase. This algebra can be conveniently represented grapically.

It is convenient to shift the field X to $X_R = X - f(0)$ and define "renormalized" couplings:

$$U_R(\vec{\phi}^2) = U_0(\vec{\phi}^2 + 1/2\pi^2) - U_0(1/2\pi^2) = \tfrac{1}{2}\mu_R^2\vec{\phi}^2 + \tfrac{1}{4}\lambda_R(\vec{\phi}^2)^2 + \tfrac{1}{6}\eta_R(\vec{\phi}^2)^3. \tag{6}$$

The phase diagram we obtain in the dimensionless parameters μ_R^2, λ_R, and η_R is schematically depicted in Fig. 1. The surface labeled by H is the second-order Heisenberg critical surface and it is attracted to the HFP. It separates between the symmetric phase (at the right of H) and the broken phase. The surface X is the first-order transition surface. For $0 \leq \eta_R < \eta_c = (4\pi)^2$, X and H are separated by a line t of tricritical points which are pointwise attracted to the line of GFP. For $\eta_R > \eta_c$ the surface X continues into a surface X' up to a line l of second-order phase transitions which meet t at the BMB end point P. The surface X' does not correspond to a breaking of the O(N) symmetry but to a discontinuty in M^2 and $\langle\vec{\phi}^2\rangle$. The line l corresponds to the critical point of a liquid-gas transition.[5]

It is now a trivial matter to compute the correlation length $\xi = M^{-1}$ of the field $\vec{\phi}$ as a function of the "bare temperature" μ_0^2. The correlation critical indices are $\nu = 1$ on the Heisenberg surface H, $\nu = \frac{1}{2}$ on the tricritical line t, and $\nu = \frac{1}{3}$ at the end point P. The index associated with the specific heat, α, has the value $\frac{2}{3}$ at P. This supports the view that the end point is very special (in particular hyperscaling is violated). These properties are clearly a result of the fact that the two lines t and l intersected at P. In particular the massless BMB dilaton is the massless excitation associated with the critical line l.

We now set up the RG analysis. Even at $N=\infty$ the RG transformation cannot be handled exactly. We shall use a truncation to the space of actions of the form (1) with an arbitrary potential $V(\vec{\Phi}^2)$. We define the transformation R_S on the space of functions V by imposing the requirement that a change in the cutoff, $\Lambda \to \Lambda/S$, if accompanied by the changes $\vec{\Phi} \to \vec{\Phi}_S$ and $V \to V_S = R_S(V)$, leaves

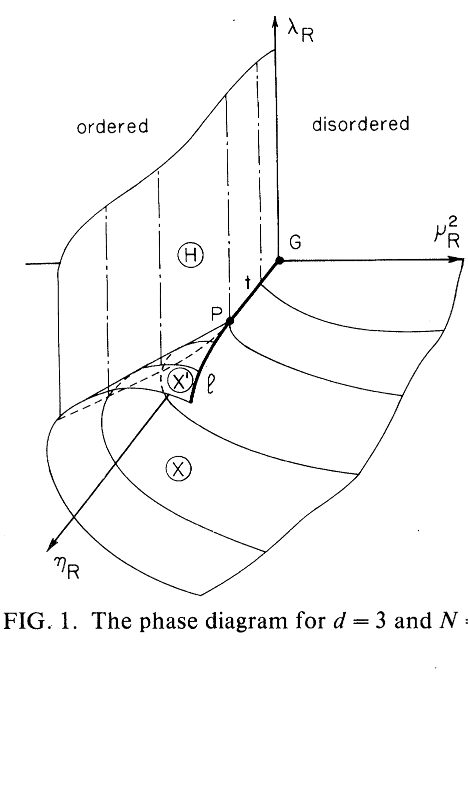

FIG. 1. The phase diagram for $d=3$ and $N=\infty$.

the effective potential unaltered:

$$V_{\rm eff}(\vec{\Phi}^2;\Lambda,V)=V_{\rm eff}(\vec{\Phi}_S^2;\Lambda/S,V_S). \qquad (7)$$

In order to keep the kinetic energy unchanged we have to pick $\vec{\Phi}_S=\vec{\Phi}$. Since at $N=\infty$ the critical exponent η is zero in perturbation theory our approximation is not implausible. In order to construct a finite continuum field theory we have to proceed as follows: Pick $\Lambda=S\Lambda_R$ with Λ_R arbitrary but finite and find a potential $V_{0,S}$ such that the limit of $R_S(V_{0,S})$ exists as $S\to\infty$ within the space of acceptable interactions. Thus the cutoff dependence is removed without giving up the explicit uv finiteness of the cutoff theory. The RG equation simplifies when we deal with the derivative of the rescaled potential $U'(\vec{\phi}^2)$: From (4a) we get

$$U_S'(\vec{\phi}^2)=S^2U'[S^{-1}\vec{\phi}^2+f((2/S^2)U_S'(\vec{\phi}^2))-S^{-1}f(2U_S'(\vec{\phi}^2))], \qquad (8)$$

which is identical to the result of Ma.[6] Introducing the inverse function of $2U_S'$, F_S (which is defined at least locally), we get

$$F_S(y)-f(y)=S[F(y/S^2)-f(y/S^2)]. \qquad (9)$$

The fixed points are obviously $F^*(y)=c\sqrt{y}+f(y)$, where c is an arbitrary parameter. As long as $-\infty\le c<0$ the function $F^*(y)$ is invertible to an analytic function $U^*(\vec{\phi}^2)$. This line of FP corresponds to the tricritical line. $c=-\infty$ corresponds to $U^{*\prime}=0$, and therefore to the GFP. The end point $c=0$ corresponds to the BMB FP. For $c>0$, $F^*(y)$ cannot be inverted into an analytic potential, except for $c=1/4\pi$, where $U^{*\prime}$ becomes analytic and monotonic. This is the $N=\infty$ HFP.

If we start with a theory in the unbroken phase we see from (4b) that there exists a $b_1\ge 0$ such that $F(b_1)=f(b_1)$. This property is preserved by the RG transformation R_S. In the vicinity of b_1 we can invert the function $G=F-f$:

$$G^{-1}(x)=b_1+\sum_2^\infty b_nx^{n-1}. \qquad (10)$$

From (9) we see that $G_S^{-1}(x)=S^2G^{-1}(x/S)$ and we obtain the critical exponents $y_n=3-n$, $n=1,2,\ldots$, associated with the nonlinear scaling fields b_n.[7] The parametrization in terms of the b_n is appropriate for a large class of potentials, including the tricritical line. The y_n are the Gaussian exponents. b_1 is in fact the square of the mass M and therefore has to have an exponent $y_1=2$. The region of interaction space which includes the HFP is best parametrized by the Taylor-series coefficients of the function $F(y)$. From (9) we obtain the critical exponents of the spherical model, $y_n=3-2n$, $n=1,2,\ldots$. Our two parametrizations overlap and hence RG flows can be followed everywhere.

We now focus on the end point of the line of tricritical FP, $c=0$. This point is on the boundary of allowed interactions. Indeed, if we start with a potential of the form (6) with $\eta_R>\eta_c$, $\mu_R^2=\lambda_R=0$, the interaction is rendered nonanalytic by the application of R_S and flows outside the allowed space of interaction for sufficiently large S. (At infinite N, even one step in the recursion involves averaging over an infinite number of degrees of freedom and might include a nonanalyticity.) When $c=0$, the scaling field b_1 depends nonanalytically on the bare parameter μ_0^2 and is not quite adequate: It is by this mechanism (nonanalyticity of the scaling fields) that the exponents become peculiar at the end point and in particular $\nu\ne 1/y_1$. The fixed-point potential U^* itself becomes nonanalytic (and therefore outside the realm of Feynman diagrams) for $c=0$, and develops a logarithmic attractive divergence at $\vec{\phi}^2=0$. This is acceptable only because the logarithm is exactly canceled by the centrifugal repulsion resulting from the integration measure. If we write $\vec{\phi}=\rho\vec{\Omega}$ with $\vec{\Omega}^2=1$, the functional integral corresponding to the fixed point becomes

$$\int D[\rho]D[\vec{\Omega}]\delta(\vec{\Omega}^2-1)\exp\{-N\int[\tfrac{1}{2}\rho^2(\partial_\mu\vec{\Omega})^2+\tfrac{1}{2}(\partial_\mu\rho)^2+W^*(\rho^2)]d^3x\}, \qquad (11)$$

where $W^*(\rho^2)$ is now analytic at $\rho^2=0$. The ρ field $[(\vec{\phi}^2)^{1/2}]$ has become a canonical field. Since N is infinite the integration over the radial components $\vec{\Omega}$ can be done and one obtains an effective potential for the field ρ which is completely flat in the range $0\le\rho\le\rho_{\max}=1/(2\pi^2)^{1/2}$. Scale invariance can be spontaneously broken; ρ is the dilaton discovered by BMB and may take any expectation value between 0 (where $M^2=\infty$) and $\rho_{\max}$ (where $M^2=0$ and scale invariance is restored). Different continuum theories correspond to these different cases.

The picture of what is happening at infinite N is now quite complete. For finite but large N the tricritical line t and the Ising line I should survive but the generic situation is that they do not meet. This would mean that the existence of the special end point P is an artifact of the large-N limit in three dimensions. That the lines t and I do not have to meet for some topological reason can be seen at infinite N in $3+\epsilon$ ($\epsilon > 0$) dimensions where the tricritical line is attracted to the GFP and its end point is by no means special. It is plausible that the situation is similar in $d=3$ for any finite N. If this is so, there is no reason to believe that a "nontrivial" $(\vec{\Phi}^2)^3_3$ theory exists for finite N. To be sure, our analysis does not rule out the possible occurrence of a nontrivial uv FP at finite N and its relationship to the BMB phenomenon. The computation of $1/N$ corrections might encounter difficulties due to the nonanalyticity at leading order. To establish the phase diagram via Monte Carlo simulations should be relatively straightforward and probably much easier than to investigate numerically the continuum limit directly.

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