Monopoles Admit Parastatistics

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It is claimed that extended monopoles of the 't Hooft–Polyakov type admit parastatistics in the sense of Finkelstein and Rubenstein.

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The Finkelstein-Rubenstein interpretation of statistics in kink field theory[1] is more akin to the interpretation used in first-quantized point-particle mechanics than that in second-quantized, algebraic, field theory. The exchange of position of two kinks introduces a change of phase of the wave function. If the phase is -1 this describes Fermi statistics, if the phase is $+1$, Bose statistics, and any other possible phases, parastatistics. (The term parastatistics originally referred to non-Abelian representations of the symmetric group but Finkelstein and Rubenstein used it in the above sense.)

In order to orient the reader an example of the way in which particle statistics can be considered to arise in point-particle mechanics is described in a language which extends to Finkelstein and Rubenstein's discussion of the statistics of kinks and the present discussion of monopoles. A system consisting of k distinguishable point particles in n dimensions has $(R^n)^k$, the Cartesian product of R^n, k times, as its configuration space. The points are ordered k-uples $(\underline{r}_1,\underline{r}_2,\ldots,\underline{r}_k)$, $\underline{r}_k \in R^n$. However, it is natural to exclude points of coincidence in which case the configuration space becomes $M_k(R^n)=(R^n)^k \backslash D_k(R^n)$ where $D_k(R^n)$ is the subset of $(R^n)^k$ such that $\underline{r}_i=\underline{r}_j$ for at least one pair $i\neq j$. If the particles are also assumed to be indistinguishable the configuration space is further restricted to the quotient $C_k(R^n)=M_k(R^n)/S_k$, where S_k is the symmetric group which acts on $M_k(R^n)$ by permuting the k-uples. That is, $C_k(R^n)$ is the space formed from $M_k(R^n)$ by identifying points which are related by permutations of the k-uples. The existence of Fermi statistics and indeed parastatistics requires that multivalued wave functions can be defined on the configuration space $C_k(R^n)$.

The physicist is very familiar with multivalued functions in complex variables, for example, the logarithm function $f(z)=\ln z=\ln|z|+i\arg z$. The function is not defined at the origin and tracing the value of the function as a point z moves in a loop, a closed path, around the origin the function changes by $2\pi i$: $f(r,\theta)=f(r,\theta+2\pi)+2\pi i$, that is, the function is multiply valued. The way in which this multiple valuedness is resolved is by either using a cut, in which case tracing the function around in a loop about the origin is disallowed, or by considering the Riemann surface, a helixlike superposition of planes, with which everyone is familiar. There are an infinite number of points on the Riemann surface, one for each integer, which correspond to a single point on the original complex plane. Thus, the original complex plane can be thought of as a quotient S/Z of the Riemann surface S by the group of integers Z. The function f is single valued on the Riemann surface and this gives rise to an infinite number of values on the complex plane by projection, one value for each integer. It is the concept of a Riemann surface which is generalizable.

In the generalization, the Riemann surface is called the covering space of the "configuration space" on which the function is defined. In the above example the configuration space is the complex plane minus the origin. The way in which the covering space can be constructed is by considering loops in the configuration space. The actual shape of the loop is immaterial. Two loops are considered equivalent, and said to be homotopic, if one can be continuously deformed into the other. This concept is again very familiar in complex variables; when considering integrals around a singularity it is not the locus of the loop that is important but what singularities it encloses and how many times it encircles them. A loop which encircles a singularity is not homotopic to a loop which does not and a loop which encircles a singularity n times, say, is not homotopic to a loop which encircles a singularity m times for m different from n. In the logarithm example the only singularity is the origin and loops are distinguished by the winding number, the number of times the origin is encircled. Thus, the homotopy classes of loops are classified by the group of integers Z, the winding number. This is the group which appears in the quotient S/Z. In general, homotopy classes are classified by a group which is called the fundamental group of the configuration space and denoted by π_1. It is the fundamental group which determines the "number" of multiple values a function defined on the configuration space may take. Another example familiar to physicists, the rotation group SO(3) which describes the configuration space of a spherical top, can be interpreted in this language. It has covering space SU(2) and fundamental group $Z/2$, the group of integers modulo 2. In consequence, it admits double-valued functions which manifest themselves in physics as half-integral spin.

For $n \geqslant 3$ the fundamental group of $C_k(R^n)$, denoted $\pi_1[C_k(R^n)]$ is the symmetric group, S_k. For quantum mechanics this multiple valuedness is realized by phase factors which form a representation of the fundamental group.[2] There are only two complex scalar representations of the permutation group: Bose statistics in which all permutations are represented by the unit 1 and Fermi statistics where a permutation is represented by ± 1 depending on whether the permutation is odd or even.

For two spatial dimensions things are very different, $\pi_1[C_k(R^n)] = \mathrm{Br}_k$, the braid group.[3] As this situation is relevant to monopoles this will be further explained. Let $\{\underline{c}_1, \underline{c}_2, \ldots, \underline{c}_k\}$ be k collinear points in R^n. A loop based at $\{\underline{c}_1, \underline{c}_2, \ldots, \underline{c}_k\}$ in $C_k(R^n)$ may be thought of as a braid on k strings with homotopic loops corresponding to equivalent braids. An example for $k=3$ is shown in Fig. 1. The braid group Br_k can be described by generators $b_1, b_2, \ldots, b_k$ where the b_i is represented by the braid in Fig. 2, an interchange of particles c_i and c_{i+1}. As can be appreciated from Fig. 2 the generators are subject to two sets of relations:

$$b_i b_j = b_j b_i, \quad 1 \leqslant i,\ j < k-1,\ |i-j| > 2\ ;$$

$$b_i b_{i+1} b_i = b_i + 1 b_i b_{i+1}, \quad i = 1, \ldots k-2\ .$$

More details can be found in Ref. 3. The scalar representations of a generator b_i are given by $\exp(i\alpha)$, $\alpha \in [0, 2\pi)$. For $\alpha = 0$ this describes bosons and for $\alpha = \pi$ fermions; other values describe anyons.[4]

This anyon behavior of point particles in two dimensions is tempered when they are given extended monopole structure. The configuration space of a general Yang-Mills-Higg's field is given by the quotient of a pair, $(C \times F)/G$, where C is a space of gauge potentials A (connections), F the configuration space of the Higg's field, and G the gauge group. In the t' Hooft–Polyakov model, the configuration space of the Higg's field F is the space of maps $R^3 \rightarrow R_H^3$ such that the sphere at infinity of physical space R^3 is mapped into the unit sphere of the iso-space R_H^3. This space of maps is denoted descriptively by $(R_H^3, S_H^2)(R^3, S_\infty^2)$. As described previously,[5] it is the boundary map $S_\infty^2 \rightarrow S_H^2$ which gives rise to monopoles. The degree of the boundary map counts the algebraic number of monopoles, a monopole contributing $+1$ and an antimonopole -1 to the degree. Thus, although monopoles live in three-dimensional space their topological structure is essentially two dimensional. It is this fact that accounts for the possible unusual statistics.

The relation between monopoles and point particles in two dimensions can be explained as follows. Consider k points on the sphere at infinity surrounded by small islands. Map the "sea" to the North pole N_H of S_H^2 and each "island" with a fixed map of degree 1 into S_H^2. The "small island" map can be extended from the boundary sphere S_∞^2 into the interior R^3 by strings (narrow tubes attached to

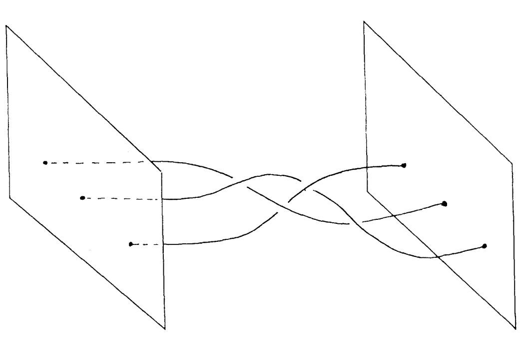

FIG. 1. An example braid.

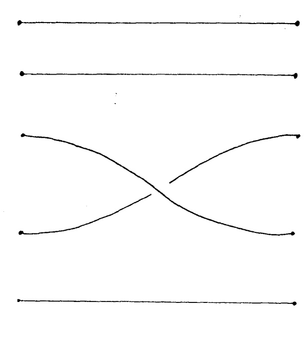

FIG. 2. A representation of a generator of Br_6.

S^2_∞ at each island). The strings are not allowed to intersect and they terminate in small spheres at chosen points in the interior. Points outside the tube, "the under sea," are mapped into the North Pole N_H of S^2_H. The value taken at points inside the tube and on the upper half of the terminating hemisphere are fixed degree-one maps given by projection from the island along lines parallel to the axis of the tube. Points on the outer hemisphere of the terminating spheres are mapped to N_H. The value of the field inside the small sphere is given by radial projection. Thus, at the center of the small sphere the value of the field is zero and this corresponds to the monopole center. In this way a k-monopole configuration can be associated with k points on the boundary sphere. Interchange of monopoles can be achieved by shrinking the string to the boundary, interchanging the position of the small islands on the boundary, and then extending the strings to their previous positions. This shows the relation between monopole interchange and the braid group of k particles on a sphere S^2_∞.

If the gauge group is assumed to be $(\mathrm{SU}(2),I)(R^3,S^2_\infty)$, that is, the space of all maps from R^3 to SU(2) such that $g(r) \in \mathrm{SU}(2)$ tends to the identity element of the group I as r tends to infinity, then using the above correspondence, it can be shown (the details of the proof involve esoteric topology and are not illuminating) that the fundamental group associated with a k-monopole configuration is $Z/2k$, the group of integers modulo $2k$. The representations of $Z/2k$ are $\exp[i\pi m/k]$ where m is an integer in the range $0 \leq m < 2k$. Thus, with the above assumptions monopoles admit parastatistics in the sense of Finkelstein and Rubenstein. The essential reason why Finkelstein and Rubenstein found that kinks did not admit parastatistics is that they are intrinsically three dimensional; monopoles, on the other hand, are intrinsically two dimensional.

Note that the word admit is to be understood in the sense of "allows." This is the same sense in which SO(3) admits double-valued representations. It means that such representations are possible but by no mean obligatory. It is not compulsory that an interchange of two monopoles in a k-monopole configuration give rise to a change of phase of the wave function by $\exp[i\pi/k]$. The equally admissible $m=0$ case gives Bose statistics and the choice $m=k$ gives Fermi statistics. Any other choice does give parastatistics. These forms of statistics are unusual in that they depend on the number of monopoles.

The use of covering spaces and fundamental groups in relation to the statistics of monopoles has been examined before.[6] In this reference it was claimed that monopoles admitted Fermi statistics. This partial result was based on an incomplete analysis of the fundamental groups of monopole configurations and does not contradict the present findings. The complete analysis reveals the richer structure which is reported herein.

Jackson[7] claimed that monopoles admit spin. This claim is based on the same assumptions of the form of the gauge group that are assumed herein. The word admit was also used in the same sense as above. Indeed, the whole tower of spins (integral and half integral) are admitted in the same way that the spherical top admits the whole tower of spins.[2] A nice pictorial indication of the existence of monopole spin has been given by Misner, Thorne, and Wheeler.[8] In Fig. 41.6, the two concentric spheres represent the boundary sphere of space and the unit sphere of isospace. The end points of the strings are representative pairs of the graph of the identity map $S^2 \rightarrow S^2_H$, which is an example of a one-monopole configuration. The frames of the figure show the unwinding of a 4π rotation of the identity map. This implies that a 2π rotation is classified by an element of the fundamental group which is at most of order two, that is, at most double-valued functions are admissible.

Although the possibility of multivalued complex functions over function spaces can be derived implicitly using arguments about fundamental groups, it is hard to visualize these functions. An example of such a double-valued function for kinks has been given.[9] One of the present authors has shown the relation of Dirac monopoles to spin.[10] This relation is more interesting than for t' Hooft–Polyakov monopoles for two reasons. Firstly, the Dirac monopole is directly associated with spinor fields; there is no tower of spins. Secondly, the argument is completely classical using only elementary electrodynamics and differential geometry. From work in progress[11] it seems as if Dirac monopoles are naturally combined using Fermi statistics; again this result is derived from classical electrodynamics and differential geometry.

The association of Dirac monopoles with magnetic charge rather than electric charge is completely arbitrary (Ref. 7, p. 251). This situation is similar to the arbitrary choice of conventional electric current flow made by Benjamin Franklin. Basing his judgement on his experiments in electrostatics he imagined electric current as a fluid flowing from one charged body to another. His investigations did not reveal what sign of charge the fluid carried but he conjectured it was positive. It is evident that, provided one is consistent and not trying to understand more fundamental structure, reliable predictions can be made with incorrect assumptions. This may be analogous to the present situation; perhaps by associating monopoles with electric charge we can achieve the (overdue) deeper understanding of spin, statistics, and the discrete nature of electric charge.

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