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N-Color Ashkin-Teller Model in Two Dimensions: Solution in the Large-N Limit

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The N-color Ashkin-Teller model is solved exactly in the large-N limit in two dimensions. The phase diagram is found. It is shown that for positive four-spin coupling the transition is first order while for negative four-spin coupling the transition is continuous and Ising type. The specific heat near the second-order transition is calculated and it is found to be finite at the transition because of large corrections to scaling. The latent heat and correlation length at the first-order transition are also calculated.

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The two-dimensional Ashkin-Teller model has attracted considerable attention in recent years. The two-color model ($N=2$) has a line of fixed points terminating at a Kosterlitz-Thouless phase transition. The model is solvable at its self-dual line through a mapping to the Baxter model.[1,2] The case $N>2$ is not as well understood. Grest and Widom[3] have performed a detailed study for weak four-spin coupling. They concluded that if the four-spin coupling g is positive the transition is first order and if $g<0$ it is second order. This conclusion resulted from a renormalization-group (RG) analysis, which showed that for $g>0$ and $N>2$ g is *marginally relevant,* and a Monte Carlo simulation for $N=3$ that indicated a first-order transition. Conversely, for $g<0$ and $N>2$ they found g to be *marginally irrelevant.*

The $N \rightarrow 0$ limit of this model represents the random-bond Ising model. In a recent paper Dotsenko and Dotsenko[4] calculated the specific heat and other quantities in perturbation theory. They showed that there are significant changes in the form of the singularities. I will show below that there are corrections to scaling also present in the $N\rightarrow\infty$ limit. Their form is, however, different in both cases.

In this note I show that the N-color Ashkin-Teller model is exactly solvable in the large-N limit. I also show that the transition is first order for $g>0$ and second order for $g<0$. Furthermore, I calculate the latent heat, the discontinuity of the magnetization, and the correlation length at the first-order transition. For $g<0$ I calculate the specific heat near the transition and show that it becomes finite as a result of strong corrections to scaling. This result is rather surprising at first sight. In four dimensions,[5-7] where the specific-heat exponent of the Ising model is zero just as in two dimensions, corrections to scaling only change its form from $\ln|t|$ to $(\ln|t|)^{1/3}$. I also show that other physically interesting quantities do not acquire corrections to scaling at $N=\infty$. Such corrections are expected to appear in first order in a $1/N$ expansion. These results will be published elsewhere.

The Hamiltonian of the N-color Ashkin-Teller model is

$$-H=\beta \sum_{\langle \vec{r},\vec{r}\,'\rangle} \sum_{\alpha=1}^{N} \sigma_\alpha(\vec{r})\sigma_\alpha(\vec{r}\,') + \frac{g}{2N} \sum_{\langle \vec{r},\vec{r}\,'\rangle} \Big[\sum_{\alpha=1}^{N} \sigma_\alpha(\vec{r})\sigma_\alpha(\vec{r}\,')\Big]^2, \tag{1}$$

where the sum runs over nearest-neighboring points $\langle \vec{r}, \vec{r}\,' \rangle$ of a square lattice of size $L \times L$. To show that simplicity is attained in the large-N limit I use the identity

$$\int_{-\infty}^{+\infty} \frac{d\phi(\vec{r},\vec{r}\,')}{(2\pi)^{1/2}} \exp\left\{ -\frac{N\phi^2(\vec{r},\vec{r}\,')}{2} + g^{1/2}\left[\sum_{\alpha=1}^{N} \sigma_\alpha(\vec{r})\sigma_\alpha(\vec{r}\,')\right]\phi(\vec{r},\vec{r}\,')\right\} = \frac{1}{\sqrt{N}} \exp\left\{\frac{g}{2N}\left[\sum_{\alpha=1}^{N}\sigma_\alpha(\vec{r})\sigma_\alpha(\vec{r}\,')\right]^2\right\}. \quad (2)$$

Thus the partition function (PF) is

$$Z = \int \prod_{\langle \vec{r},\vec{r}\,'\rangle} \frac{d\phi(\vec{r},\vec{r}\,')}{(2\pi)^{1/2}} \sum_{\{\sigma_\alpha(\vec{r})\}} \exp\{-S[\phi,\sigma]\}, \quad (3)$$

with

$$S[\phi,\sigma] = \sum_{\langle \vec{r},\vec{r}\,'\rangle} \{\tfrac{1}{2}N\phi^2(\vec{r},\vec{r}\,') + [\beta + g^{1/2}\phi(\vec{r},\vec{r}\,')] \sum_{\alpha=1}^{N} \sigma_\alpha(\vec{r})\sigma_\alpha(\vec{r}\,')\}. \quad (4)$$

After integrating out the Ising spins one finds

$$Z = \int \exp[-\sum_{\langle \vec{r},\vec{r}\,'\rangle} \tfrac{1}{2}N\phi^2(\vec{r},\vec{r}\,')][Z_{\rm I}\{\beta + g^{1/2}\phi(\vec{r},\vec{r}\,')\}]^N \prod_{\langle r,r'\rangle} \frac{d\phi(\vec{r},\vec{r}\,')}{(2\pi)^{1/2}}, \quad (5)$$

where $Z_{\rm I}\{k(r,r')\}$ is the PF of an Ising model with bond strengths $k(r,r') = \beta + g^{1/2}\phi(r,r')$.

In two dimensions the PF of an Ising model with *arbitrary* bonds has the form[8]

$$Z_{\rm I}\{k(\vec{r},\vec{r}\,')\} = [\det \hat{M}\{k(\vec{r},\vec{r}\,')\}]^{1/2}[\prod_{\langle \vec{r},\vec{r}\,'\rangle} \cosh k(\vec{r},\vec{r}\,')] \equiv e^{-F_{\rm I}}, \quad (6)$$

where $\hat{M}$ is a $4L \times 4L$ matrix.[4,9] Thus the PF is

$$Z = \int \prod_{\langle \vec{r},\vec{r}\,'\rangle} \frac{d\phi(\vec{r},\vec{r}\,')}{(2\pi)^{1/2}} \exp[-NS\{\phi\}], \quad (7)$$

where

$$S[\phi] = \sum_{\langle r,r'\rangle} \{\tfrac{1}{2}\phi^2(r,r') - \ln\cosh[\beta + g^{1/2}\phi(\vec{r},\vec{r}\,')]\} - \tfrac{1}{2}\ln\det\hat{M}\{\beta + g^{1/2}\phi(\vec{r},\vec{r}\,')\} \quad (8)$$

is the effective action.

The form of Eq. (7) suggests a saddle-point approximation. In this approach all one has to do is to find the field $\phi(\vec{r},\vec{r}\,')$ which extremizes $S[\phi]$. The thermodynamically stable state will be that of lowest energy. Since the solution of the saddle point equation (SPE) will be uniform (and isotropic) one simpy has to find the extrema of $S[\phi]$ for constant $\phi(\vec{r},\vec{r}\,') = \bar{\phi}$. Introducing $x = \tanh(\beta + g^{1/2}\bar{\phi})$ and using the expression for $\det\hat{M}$ from the Onsager solution,[8] I find

$$v(\bar{\phi}) \equiv \frac{S[\bar{\phi}]}{L^2} = \bar{\phi}^2 + \ln(1-x)^2 - \frac{1}{2}\int \frac{d^2p}{(2\pi)^2} \ln[(1+x^2)^2 - 2x(1-x^2)(\cos p_1 + \cos p_2)], \quad (9)$$

where the integral is restricted to the first Brillouin zone.

The SPE is found by minimizing $v(\bar{\phi})$. The result is

$$\tanh^{-1}x = \beta + gF(x), \quad (10)$$

with

$$F(x) = \frac{1}{4}\left\{x + \frac{1}{x} + \frac{2}{\pi}\frac{kK(k)}{x}\left[x(1+x^2) - \frac{(1-3x^2)}{k}\right]\right\}, \quad (11)$$

where $k=4x(1-x^2)/(1+x^2)^2$ and $K(k)$ is the complete elliptic integral of the first kind. Once a solution of Eq. (10) is found one concludes that, at $N=\infty$, all the colors behave like an effective Ising model with effective coupling $\bar{K}=\beta+g^{1/2}\bar{\phi}$. If $\bar{K}$ happens to equal the Onsager value $K_0=\frac{1}{2}\ln(\sqrt{2}+1)$ the correlation length of the system would be finite. It turns out that $\bar{K}=K_0$ is not a low-energy solution *unless* $g<0$. For $g>0$ if $\bar{K}=K_0$ is a solution there are always two other solutions K_1 and K_2 such that $K_1<K_0<K_2$ with lower energy. In fact there is a curve in the (β,g) plane on which K_1 and K_2 have the same energy. Since $K_1<K_0$ the system is disordered and the correlation length is finite. Conversely for $K_2>K_0$ the system orders. Hence this curve represents the phase boundary of a *first-order* transition. For $g<0$ there is a curve in the (β,g) plane in which $\bar{K}=K_0$ is the *only* solution and this curve is the phase boundary of a *second-order* transition. For $g<0$ it coincides with the straight line $\beta+g\sqrt{2}/2=K_0$, while for $g>0$ they coincide if g is very small. It is worth noting that at $N=\infty$ this system has only two phases. Grest and Widom[3] find a richer phase diagram. The extra phases disappear at large N, like $1/N$.

It is instructive to solve Eq. (10) for $\bar{K}\sim K_0$ (or $x\sim x_0=\sqrt{2}-1$). By setting $x=x_0-\delta$, Eq. (10) now reads

$$0\simeq s+A\delta+B\delta\ln|\delta|, \tag{12}$$

where

$$s=\beta+g\frac{\sqrt{2}}{2}-k_0,\quad A=\left(\frac{\sqrt{2}+1}{2}\right)\left\{1+g-\frac{4}{\pi}g\ln[2\sqrt{2}(\sqrt{2}-1)]\right\},\quad B=\frac{4}{\pi}(\sqrt{2}+1)g.$$

Setting $s=0$ (i.e., on the phase boundary) I find the solutions

$$\delta_0=0, \tag{13a}$$

$$|\delta_0|=e^{-A/B}=[2\sqrt{2}(\sqrt{2}-1)e^{-\pi/4}]^{1/2}e^{-\pi/8g}\quad\text{(double root)}. \tag{13b}$$

For $g>0$ the lowest-energy state is (13b) which is a double root. For $g<0$ the only solution is $\delta=0$ (i.e., $x=\sqrt{2}-1$). If $s\neq0$ but small, one finds

$$\delta\simeq\begin{cases}\delta_0-\dfrac{\pi(\sqrt{2}-1)}{4g}s, & s<0,\\[2mm] -\delta_0-\dfrac{\pi(\sqrt{2}-1)}{4g}s, & s>0,\end{cases} \tag{14a}$$

$(g>0)$ and

$$\delta\simeq\frac{\pi(\sqrt{2}-1)}{4|g|}\,\frac{s}{\ln|s/4\pi^{-1}(\sqrt{2}+1)g\delta_0|} \tag{14b}$$

$(g<0)$. With these results on hand we can read off the following physical properties.

(i) $g>0$: The transition is first order. The value of x at the phase boundary determines the correlation length at the transition: It is just the correlation length of an Ising model with $x=x_0-\delta$. For $g\to0$, $\delta\to0$ and the correlation length is[10]

$$\xi\simeq\frac{1}{4|K-K_0|}=\left(\frac{\sqrt{2}-1}{2}\right)\frac{1}{|\delta_0|}\simeq\left(\frac{\sqrt{2}-1}{4}\right)^{1/2}\exp\left[\frac{\pi}{8}\left(1+\frac{1}{g}\right)\right]. \tag{15}$$

The latent heat Q can be calculated by computing the discontinuity of the slope of the free energy across the transition. The result is

$$Q\simeq[8\sqrt{2}(\sqrt{2}+1)]^{1/2}g^{-1}\exp[-(\pi/8)(1/g+1)]\quad(K\to0). \tag{16}$$

Both Q and ξ satisfy the RG equation of Grest and Widom,[3]

$$\frac{dK_4}{dl}=(8/\pi)(N-2)K_4^2, \tag{17}$$

if one makes the identification $N^{1/2}K_4=g$. This expression suggests that these formulas may become exact by replacing $g^2 \rightarrow (N-2)K_4^2$ as dictated by scaling. The same type of calculation indicates that at $N=\infty$ the magnetization M_c at the transition jumps to zero like $|\delta_0|^{1/8}$.

(ii) $g<0$: The transition is second order and Ising type. The correlation function *at* $N=\infty$ at the phase boundary diverges like $1/R^{1/4}$ and the magnetization vanishes like $|\beta-\beta_c(g)|^{1/8}$. Since g is marginally irrelevant these results are expected to acquire corrections to scaling already *at* $N=\infty$. It can be calculated by differentiating the free energy twice. The result is ($s \rightarrow 0$, g fixed)

$$C_s \sim \frac{2}{|g|}\left[1+\left(\frac{8}{\pi}\sqrt{2}(\sqrt{2}+1)|g|\ln\left|\frac{|s|\exp(1+\pi/8)(\pi/4)(\sqrt{2}-1)}{\ln(|s|/[2\sqrt{2}(\sqrt{2}-1)]^{1/2}|g|)}\right|\right)^{-1}+\ldots\right]. \tag{18}$$

Thus the specific heat no longer diverges at the transition if $g<0$. Instead the specific heat has a cusp at the transition since its derivative $\partial C_s/\partial s$ diverges like $|s\ln^2|s||^{-1}$. Naturally, if $g\rightarrow 0$ (s fixed) one recovers the logarithmic divergence of the two-dimensional Ising model. Note that Eq. (18) is valid for all $g<0$ as long as $s\rightarrow 0$.

In conclusion, I solved the N-color Ashkin-Teller model in the $N=\infty$ limit. I showed that the transition at the decoupling point is a tricritical point. Explicit formulas are given for the latent heat, magnetization jump, and correlation length at the first-order transition ($g>0$) as well as the specific heat and other quantities at the second-order transition ($g<0$).

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