Fractal Dimensionality of Brownian Motion

Although the molecules of a fluid obey well-defined equations of motion, the rapidly changing local environment of each molecule results in its following a trajectory that, to the macroscopic observer, appears as a random walk in space—the phenomenon of Brownian motion. The random walk has fractal dimensionality[1] $D=2$.

A metrical definition of D for continuous curves is based on the notion of length measurements conducted on different scales[1]: A pair of dividers is opened to a width η and the "length" of the curve measured by using the dividers to step through a sequence of points on the curve separated by distance η. The number of such points is $\eta^{-D'}$ where, if the curve exhibits self-similarity (or scale invariance) over a range of η, $D'=D$. The total length of the curve is $L(\eta)=\eta^{1-D'}$. A recent Letter[2] posited the claim that the value of D' for Brownian motion is not 2, but 1.65, based on molecular dynamics simulation of a Lennard-Jones liquid.

Were such a result to be upheld it would seriously detract from the elegance of the fractal point of view. To settle the issue we have carried out a computer study of a similar kind, but this time using a hard-sphere fluid. The system consisted of 1372 spheres of unit diameter; the computational techniques required to follow the dynamics are described elsewhere.[3]

A potential source of error in the numerical estimation of D' is inadequate convergence due to molecular trajectories of insufficient length. To minimize this possibility the trajectories used here are 10^4 steps in length; to produce such long trajectories the system must be followed for a total of 7×10^6 collisions. The complete trajectories (i.e., the coordinates at each collision) of sixteen of the molecules (after allowing for equilibration) provide the data on which the measurements are based. The time periods covered by the simulations are 12–28 times (depending on density) that of the earlier work[2] and, consequently, whatever self-similarity is inherent in the trajectories should be even more apparent here; furthermore, the analysis of several trajectories rather than a single one should improve the reliability of the results.

The measured trajectory lengths (allowing for periodic boundaries) are shown in Fig. 1; the gradient of the linear portion is an estimate of $1-D'$. The straight lines are included to serve as visual guides and indicate the expected slope if indeed $D'=2$. The bars show the rms spread of the $L(\eta)$ values; the spread is essentially independent of η, but is distorted by the log scale.

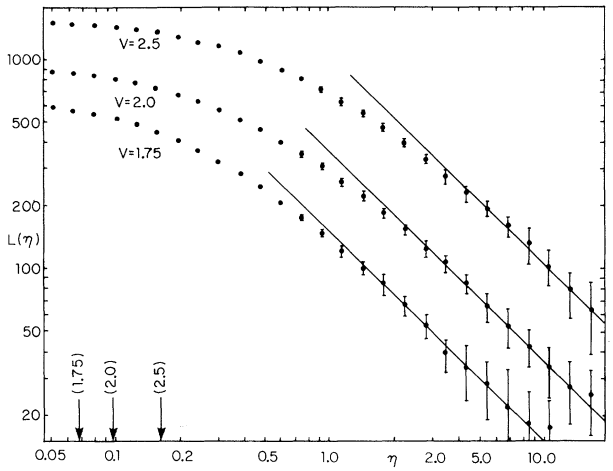

FIG. 1. Log-log plot of trajectory length $L(\eta)$ vs scale length η for various values of reduced volume (v). The straight lines show the expected behavior if $D'=2$. The mean-free-path lengths for each v are indicated by arrows.

The outstanding feature of the graphs is that, for each density, they converge to linearity with $D'=2$; there is not the slightest indication of linear behavior with any other value of D'. Thus the original prediction[1] $D'=D=2$ is vindicated. The suggestion that D' could serve as a state variable[2] is of course no longer tenable.

The figure also reveals how a result such as $D'=1.65$ might be deduced from the data. The rate of convergence to linearity is surprisingly slow; linearity first appears when η is typically 25 (±5) times the mean-free-path length. Since the result $D'=1.65$ was based on a trajectory that terminated after only 30 mean-free-path lengths it is hardly surprising that the correct limiting behavior was missed.

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