Intermittency and Solitons in the Driven Dissipative Nonlinear Schrödinger Equation

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The cubic nonlinear Schrödinger equation, in the presence of driving and Landau damping, is studied numerically. As the pump intensity is increased, the system exhibits a transition from intermittency to a two-torus to chaos. The laminar phase of the intermittency is also a two-torus motion which corresponds in physical space to two identical solitons of amplitude determined by a power-balance equation.

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In this Letter, we present new results of a numerical study of the transition to chaos in a driven, damped, cubicly nonlinear Schrödinger (NLS) equation. We believe that this work bears on a variety of important relationships pervading current studies of stochasticity, such as the contrasting behavior between dissipative and Hamiltonian systems, the differences between systems with a few degress of freedom and with many degrees of freedom [i.e., partial differential equations (PDE's)], and the relationship between highly coherent nonlinear excitations, such as solitons or solitonlike structures, and states which exhibit intrinsic chaos. We elucidate these differences as follows.

The undamped NLS (with appropriate initial and boundary conditions)[1,2] possesses, among its solutions, a finite-energy bound state of N solitons of incommensurate amplitudes,[2] whose space trajectories lie on an N torus.[3,4] If we imagine a sequence of progressively increasing total energies for such a state, then N increases, and arbitrarily complicated motions can be built up. Such an evolution would be in accord with Landau's[5] conjectures on transitions to turbulence. However, our present analysis of the driven, damped NLS shows (as do other studies[6,7] of PDE's) a transition from a two-torus directly into chaos, with increased driver strength. This result is not seen in studies of a highly truncated[8] driven, damped NLS equation, which exhibits a different route to chaos—the Feigenbaum sequence of period-doubling bifurcations. Of perhaps more importance, we find, for lower driver strengths, that the regular (two-torus) motion is punctuated by intermittent bursts of chaos.

Whereas recent studies[9] of the NLS found stochastic behavior in a few parameters which characterize an *assumed* soliton in the limit of weak driving and damping, we do not impose such a weakness condition, and find that an intermittent or permanent transition to chaos is accompanied by the complete destruction of a solitonlike state.

The equation that we study is

$$i[A_t-\int\Gamma(x-x')A(x',t)\,dx']+\tfrac{3}{2}A_{xx}+|A|^2A=0, \tag{1a}$$

where the Fourier transform of the dissipation Γ is

$$\Gamma_k=g[\delta_{-\Delta k,k}+\delta_{0,k}+\delta_{\Delta k,k}]-\gamma_k, \tag{1b}$$

$$\gamma_k=(\pi/8e^3)^{1/2}|k|^{-3}\exp(-k^{-2}/2). \tag{1c}$$

In plasma physics this equation constitutes one idealized model of externally excited Langmuir wave turbulence, in which A is the complex amplitude of the wave field $\mathrm{Re}[\bar{A}\exp(-i\omega_p\bar{t})]$. The dimensionless variables, t,x,k,A, are related to the dimensional variables $\bar{t},\bar{x},\bar{k},\bar{A}$ by $t=\omega_p\bar{t}$, $x=k_D\bar{x}$, $k=\bar{k}/k_D$, and $A^2=\bar{A}^2/32\pi n\theta_e$, where ω_p is the electron plasma frequency, k_D the Debye wave number, and n and θ_e the electron density and temperature. The Landau damping,[10] γ_k, becomes strong as k increases. The driver strength is measured by the (stochasticity) parameter g. Only long-wavelength modes are driven. The cubic nonlinearity in Eq. (1) corresponds to an adiabatic ion-density response to the ponderomotive force of the Langmuir waves. Such a response might be appropriate to a weakly driven plasma in which ion-acoustic waves are heavily damped (i.e., $T_i\approx T_e$). A more general ion response is described by the driven, dissipative Zakharov equation.[7]

We have solved Eq. (1) subject to periodic boundary conditions. The complex amplitude $A(k,t)$ is expanded into the Fourier series,

$$A(x,t)=(\Delta k/2\pi)\sum_{n=-63}^{63}a_n(t)e^{-in\Delta kx}.$$

The grid spacing, Δk, is set equal to 3.33×10^{-2}, in our units. The initial condition is a flat spectrum of random-phased "noise," of order of magnitude 3×10^{-5}. Several derived quantities that we con-

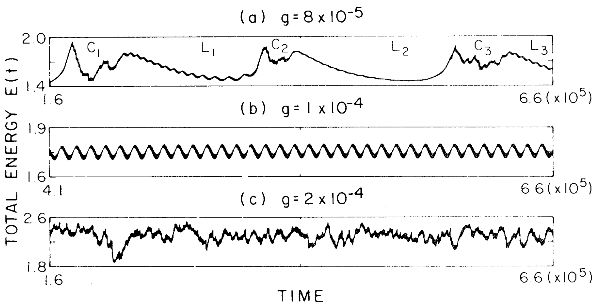

FIG. 1. Energy of the system, as a function of time, for three different values of g. L_i denotes a laminar phase and C_i a chaotic phase.

sider here are the energy, $E(t) = (\Delta k/2\pi) \times \sum_k |a_k(t)|^2$, the dissipation spectrum, $D(k,t) = \gamma(k)|a_k(t)|^2$, the rate of power loss, $P_{\text{out}}(t) = (\Delta k/2\pi) \sum_k D(k,t)$, and the rate of power input, $P_{\text{in}} = g(\Delta k/2\pi) \sum_{k=\pm\Delta k,0} |a_k(t)|^2$.

The total energy of the system, as a function of time, is displayed in Fig. 1 for three different values of g. For g between 0.5×10^{-5} and 0.94×10^{-4}, we observe the intermittent behavior of Fig. 1(a). After a sudden increase of energy, there occurs a short "burst" of chaos (labeled C_1), confirmed by positive local Lyapunov number. This is followed by a regular motion (laminar phase, labeled L_1). Many such sequences of chaotic bursts and regular motions are observed. At the critical value, $g_0 = 0.94\times 10^{-4}$, this intermittency is completely replaced by a persistent laminar motion [Fig. 1(b)]. For $g \geq g_1 = 1.2\times 10^{-4}$ we observe a permanently irregular behavior as depicted in Fig. 1(c).

Trajectories in a reduced phase space are studied by making use of reduced maps (defined in the caption of Fig. 2) for each dynamic regime. In the intermittent regime, the map of Fig. 2(a) [encompassing both laminar (L_2) and burst phases (C_3) of Fig. 1(a)] displays a closed curve during the laminar phase and the scattered dots during a burst. This suggests that the intermittent-phase dynamics observed here consists of two weakly attracting solutions, one of which appears to be a two-torus motion, and the other a chaotic motion. The return map [Fig. 2(a)] taken only from the torus depicts two areas of accumulation of points along a diagonal, reminiscent of studies of intermittency for the Belousof-Zahbotinsky reaction.[11] Figure 2(b) illustrates a two-torus found for $g \gtrsim g_0$. The wispy "thickness" becomes more pronounced as $g \to g_1$, and we believe that it may be a strange attractor.

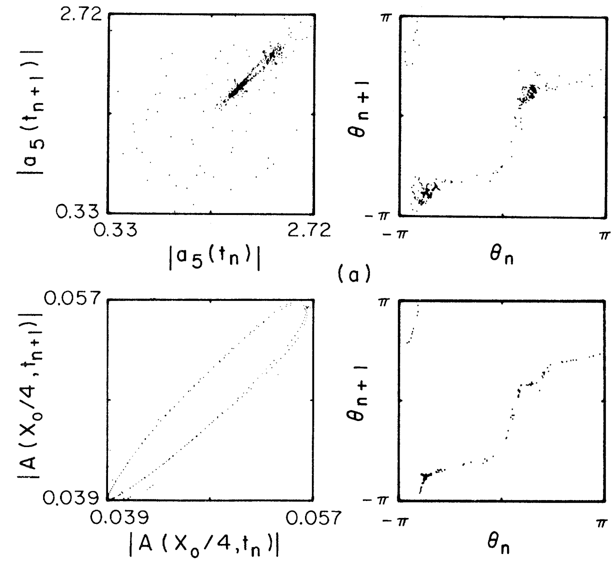

FIG. 2. (a) Intermittent regime, $g = 8\times 10^{-5}$: The left-hand map is constructed by plotting $|a_5(t_n)|$ vs $|a_5(t_{n+1})|$. t_n is the time whenever $|a_7(t_n)| = 0.98$ and $d|a_7|/dt > 0$ during $4.1\times 10^{+5} \leq t \leq 5.8\times 10^{+5}$. To create the right-hand map, each successive point appearing only on the torus section in the left-hand map is assigned an angle θ_n relative to the torus center. The return map is constructed by plotting θ_n vs θ_{n+1}. (b) Two-torus regime, $g = 1\times 10^{-4}$: The left-hand map is made by plotting $|A(x_0/4,t_n)|$ vs $|A(x_0/4,t_{n+1})|$. $x_0 = \pi/\Delta k$, and t_n is the time defined by $|A(x_0,t_n)| = 0.047$ and $d|A|/dt > 0$ in the interval $4.1\times 10^{+5} \leq t \leq 6.6\times 10^{+5}$. The right-hand figure is the corresponding return map.

There is no evidence of prior period-doubling bifurcations. The power spectrum (not shown) contains sharp lines corresponding to the two-torus. As $g \to g_1$ the valleys between these lines fill in continuously with noise.

Figure 3 illustrates the behavior of the solution $|A(x,t)|$ in real space, as a function of time, for the weakly driven case, $g = 8\times 10^{-5}$, which exhibits intermittency. Figure 3(a) shows two solitonlike structures of equal amplitude, while Fig. 3(b) demonstrates the destruction of these solitons during an intermittent chaotic burst.

Much can be learned about the energetics, spectra, and characteristic scale sizes during the laminar interval, by making the heuristic assumption that the observed saturated solitonlike structures have the form of two noninteracting soliton solutions to the undriven, undamped (i.e., Hamiltonian) NLS equation, namely,

$$A_s(x) = A_0[\operatorname{sech}(x - x_0/2)/l - \operatorname{sech}(x + x_0/2)/l]e^{i\Omega t}, \tag{2}$$

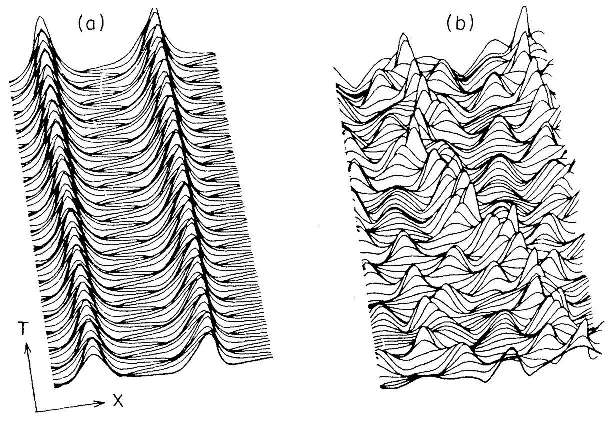

FIG. 3. (a) Two solitons appearing during a laminar phase; (b) chaotic structure during an intermittent burst.

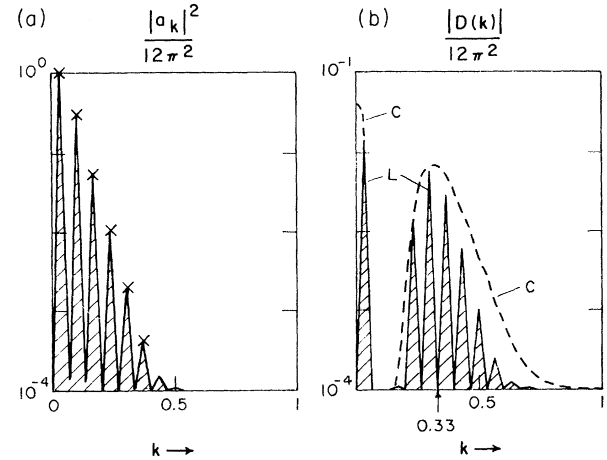

FIG. 4. (a) Energy spectrum in laminar phase; crosses are from theory [Eq. (3)]. (b) Dissipation/injection spectra (L, laminar phase; C, chaotic phase).

where $l=\sqrt{3}/A_0$, $\Omega=A_0^2/2$, and $x_0=\pi/\Delta k$ (one-half of the spatial domain). We have chosen a state of odd parity, as indicated by our numerical computations [the displayed $|A(x,t)|$ cannot, of course, show the parity]. The energy spectrum corresponding to the soliton state[12] of Eq. (2) is

$$|a_k^{\mathrm{sol}}|^2=12\pi^2\sin^2(kx_0/2)\,\mathrm{sech}^2(k\alpha/2), \tag{3}$$

where $\alpha\equiv\sqrt{3}\pi/A_0$. The interference function, $\sin^2(kx_0/2)$, oscillates between 0 and 1 for each grid point increment of $k=n\Delta k$, and may be replaced by its average value in appropriate sums over k. The total energy in the soliton spectrum is $E=4\sqrt{3}A_0$. The power input is

$$P_{\mathrm{in}}=12\pi\Delta kg\,\mathrm{sech}^2(\Delta k\alpha/2). \tag{4}$$

The (dissipated) power out may be expressed as

$$P_{\mathrm{out}}(x)\approx(6\pi^{3/2}/\sqrt{8}e^3)\int_0^\infty dk\,k^{-3}e^{-k^{-2}/2}\mathrm{sech}^2(\alpha k/2). \tag{5}$$

The integral may be evaluated asymptotically as $\alpha\to\infty$, by the method of steepest descent. The integrand has a maximum (of the dissipation spectrum) at $k_{\mathrm{diss}}\equiv\alpha^{-1/3}$, and $\lim_{\alpha\to\infty}P_{\mathrm{out}}(\alpha)=(12\times\pi^2/\sqrt{3}e^3)\alpha^{1/3}\exp(-\frac{3}{2}\alpha^{2/3})$.

From an approximate power balance, $P_{\mathrm{in}}\approx P_{\mathrm{out}}$, we obtain, to an adequate accuracy, the result $\alpha\approx[\frac{2}{3}\ln(\Delta kg)^{-1}]^{3/2}$, which gives us a prediction for the soliton amplitude in terms of the driver strength g. For the case of Fig. 3(a), $g=8\times10^{-5}$, and we predict $\alpha\approx25.05$, or $A_0=0.213$. This is within a few percent of the measured value (0.217), and predicts a total energy, $E_s=1.5$, consistent with Fig. 1(a).

With this value of A_0, the energy spectrum $|a_k^{\mathrm{sol}}|^2$ of Eq. (3) is now seen to fit the measured spectrum all the way from $k=0$ to $k\approx k_{\mathrm{diss}}$, a range over which $|a_k^{\mathrm{sol}}|^2$ varies by more than three orders of magnitude [Fig. 4(a)]. Deeper into the dissipation range the soliton spectrum decreases exponentially and is overwhelmed, in the numerical solution, by a nonsoliton spectral contribution at high k. However, it is notable that the measured dissipation spectrum has the form given by $\gamma_k|a_k^{\mathrm{sol}}|^2$ and peaks at the predicted value $k_{\mathrm{diss}}=(A_0/\sqrt{3}\pi)^{1/3}\approx0.34$ [Fig. 4(b)].

Each chaotic burst is preceded by a steeply rising fluctuation in energy, $E(t)$, as seen in Fig. 1(a). The two-soliton state loses stability, and the injection and dissipation spectrum is observed to increase significantly [Fig. 4(b)]. When the chaos subsides, the two-soliton state reappears intact.

We next consider the two frequencies of motion around the two-torus during the soliton-dominated laminar motions observed for $g=0.8\times10^{-4}$ and 10^{-4}. One frequency is measured to be of the same order as $\Omega\equiv A_0^2/2$ in Eq. (2), although the precise manner in which this phase modulation becomes the soliton amplitude modulation visible in Fig. 3(a) is not understood. The second frequency appears to correspond to an observed slow oscillation or drift in the soliton centroid position [x_0 in Eq. (2)].

In summary, we conclude that the motion of $|A|$ has undergone successive changes in state from intermittency to a two-torus to chaos. Intermittency as a root to chaos has been discussed by Pomeau and Manneville[13] for systems with a few degrees of freedom, and has also been observed in PDE's.[7] The intermittency observed here, however, is followed (as q is increased) by a two-torus motion [Fig. 1(b)], which loses its stability to permanent chaos [Fig. 1(c)]. We have shown that the measured spectrum in the laminar stage can be understood analytically in terms of the spectrum of an antisymmetric two-soliton state, even though this is a driven, damped system. The amplitude of the solitons is correctly set by the driver strength g, when an approximate power balance is assumed. The soliton spectrum reproduces the measured spectrum between the injection and dissipation wave numbers, a range over which it varies by over three orders of magnitude, and accurately determines the wave number, $k_{\rm diss}$, of maximum dissipation.

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