Laser Lorenz Equations with a Time-Dependent Parameter

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We study the stability of the zero-intensity solution of the semiclassical laser equations when the pump parameter increases linearly in time. Equivalently, we study the stability of the trivial fixed point of the Lorenz equations when the Rayleigh number is increased linearly in time. When the time-dependent parameter varies slowly, the stability domain is greatly increased with respect to the stability domain derived under stationary conditions. This corresponds to a dynamical stabilization of an unstable stationary solution.

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In many experimental studies of instabilities in lasers and in lasers with a saturable absorber, the control parameter is slowly varied in time.[1,2] In a recent study[3] we have shown that this time dependence may induce dramatic changes in the bifurcation diagram derived under the assumption of a stationary control parameter. However, the complexity of the model prevented a study beyond a perturbative approach. Therefore, we have considered a simpler problem, i.e., the semiclassical laser model in the mean-field limit with homogeneously broadened two-level atoms in a resonant single-mode cavity. This model is described by the following equations:

$$E_t = -E + Av, \quad v_t = d(-v + EF),$$
$$F_t = d_{\parallel}(-F + 1 - Ev), \tag{1}$$

where E, v, and F are the reduced field amplitude, atomic polarization, and inversion, respectively. The pump parameter A is the control (i.e., bifurcation) parameter. The two decay rates d and $d_{\parallel}$ are the polarization and atomic-inversion decay rates, respectively, measured in units of the cavity decay rate. The change of variables

$$E = \frac{x}{b^{1/2}}, \quad v = \frac{y}{Rb^{1/2}}, \quad F = 1 - \frac{z}{R},$$
$$R = A, \quad \sigma = d^{-1}, \quad b = d_{\parallel}/d, \quad \tau = td,$$

transforms Eqs. (1) into the Lorenz equations.[4] Hence A can be freely interpreted as the reduced Rayleigh number or the pump parameter and in both cases it is the relevant control parameter. The main difference is that R has to be positive whereas A can be negative as well.

We shall analyze the stability of the trivial solution

$$E = v = F - 1 = 0 \tag{2}$$

when (1) is solved with the initial condition

$$E(0) = E_0 << 1, \quad v(0) = 0, \quad F(0) = 1, \tag{3}$$

corresponding to a small perturbation of (2). Linearizing (1) around the trivial solution (2) yields

$$E_t = -E + vA,$$
$$v_t = d(-v + E), \tag{4}$$

with $E(0) = E_0 << 1$ and $v(0) = 0$. When A is constant, a linear stability analysis of the trivial solution $E = v = 0$ of (4) leads to the characteristic equation

$$\lambda^2 + \lambda(d+1) + d(1-A) = 0. \tag{5}$$

Hence, the trivial solution is stable if and only if $A \leqslant 1$.

Suppose now that A varies slowly in time, i.e., $A = A(\epsilon t) \equiv A(t')$. An important feature is that even if A is time dependent, the trivial solution (2) remains an exact solution of (1), and therefore we can still use (4). The solution of (4) for E can be written in the form

$$E = \sum_{j=1,2} c_j \exp\left[\frac{1}{\epsilon}\int_0^{t'} g_j(s,\epsilon)\, ds\right],$$

where the g_j are the two solutions of

$$g^2 + g(d+1) + d(1-A) = \epsilon[(\dot{A}/A)(g+1) - \dot{g}], \tag{6}$$

and the dot indicates a derivative with respect to t'. Equation (6) is solved by a series in powers of ϵ. To dominant order we obtain $g_0^2 + g_0(d+1) +$

$d(1-A)=0$, which is exactly the characteristic equation (5) except that now the time dependence of A induces a time dependence of g_0. Consequently, the solution E will become unstable when

$$\int_0^{t^*} g_0(s)\,ds=0. \qquad (7)$$

Let $\bar{t}$ be the time at which the stationary bifurcation is reached, i.e., $A(\bar{t})=1$. We can express the dynamical stability condition (7) as

$$\int_0^{\bar{t}} g_0(s)\,ds=-\int_{\bar{t}}^{t^*} g_0(s)\,ds. \qquad (8)$$

This equality expresses a balance between the "accumulated stability" from 0 to $\bar{t}$ (where $g_0 \leq 0$) and the "accumulated instability" between $\bar{t}$ and t^* (where $g_0>0$). In contradistinction with the condition $A=1$ which is a local property resulting from the behavior of the characteristic root at the bifurcation point, we now have a nonlocal condition: From (8) we see that t^* will depend on the rates of damping and divergence during the whole interval $0 \leqslant t \leqslant t^*$. Furthermore, we necessarily have the inequality $t^*>\bar{t}$, implying a delay of the dynamical bifurcation. The explicit solution of (7) yields

$$D(t^*)=\frac{A(t^*)-A(\bar{t})}{A(\bar{t})-A(0)}=\frac{1}{8\alpha}\{4\alpha-3+4[-3\alpha^2+19.5\alpha-11.4375+12(1-\alpha)^{3/2}]^{1/2}\}=\phi(\alpha) \qquad (9)$$

where $\alpha=4d[1-A(0)]/(d+1)^2$. When $\alpha \geqslant 1$, the solution of (6) is no longer an analytic function of ϵ and the relation (9) becomes invalid. The function $\phi(\alpha)$ varies from 1 (near $\alpha=0$) to $\frac{5}{4}$ (near $\alpha=1$). Furthermore, it is independent of the rate at which A varies, for sufficiently small ϵ; it depends on a single variable which combines the remaining two parameters, d and $A(0)$. For the class of initial values $A(0)$ near the steady bifurcation, $\phi(\alpha) \simeq 1$ and therefore

$$A(t^*)-A(\bar{t}) \simeq A(\bar{t})-A(0).$$

A similar result was obtained for a laser with saturable absorber.

The previous discussion offered an insight into the mechanism of the dynamical bifurcation, but it is restricted by the fact that A was supposed to vary very slowly and the initial condition was limited by $\alpha<1$. In this problem we can bypass these limitations by constructing the exact solution of (4). Let $A(t)=A(0)+bt$ with arbitrary sweeping rate $0<b<\infty$. We transform (4) into a closed second-order differential equation for v whose solution is

$$v(t)=E_0\pi b^{-1/3}d^{2/3}\exp[-\tfrac{1}{2}t(d+1)][\mathrm{Ai}(x_0)\mathrm{Bi}(x)-\mathrm{Ai}(x)\mathrm{Bi}(x_0)], \qquad (10)$$

in terms of the Airy functions of argument

$$x=x(t)=(bd)^{-2/3}\{A(t)d+[(d-1)/2]^2\},$$

and $x_0=x(0)$. We now analyze the solution (10) in the limit $b \to 0$ and therefore $x \to +\infty$. When $\alpha<1$, we have $x_0 \to -\infty$ and

$$v(t) \sim \exp\{\tfrac{2}{3}x^{3/2}-\tfrac{1}{2}t(d+1)-\tfrac{2}{3}x_0^{3/2}\}. \qquad (11)$$

Hence, $v(t)$ will begin to diverge when $x=x^*=x(t^*)$ given by

$$\tfrac{2}{3}(x^*)^{3/2}-\tfrac{1}{2}t^*(d+1)-\tfrac{2}{3}(x_0)^{3/2}=0. \qquad (12)$$

It is easy to verify that (12) is identical to the condition (7) and therefore leads to (9). For $\alpha=1$, we have $x_0=0$ and

$$v(t) \sim \exp\{t[\tfrac{2}{3}(bdt)^{1/2}-(d+1)/2]\}. \qquad (13)$$

Thus the critical time t^* is now given by

$$bdt^*=[\tfrac{3}{4}(d+1)]^2, \qquad (14)$$

which leads to

$$D(t^*)=\tfrac{5}{4}. \qquad (15)$$

Finally, when $\alpha>1$, the parameter x_0 diverges to $-\infty$ and the asymptotic expansion of (10) becomes

$$v(t) \sim \exp\{\tfrac{2}{3}x^{3/2}-t(d+1)/2\}, \qquad (16)$$

provided that $\mathrm{Ai}(x_0) \neq 0$. The critical time t^* is therefore given by

$$\tfrac{2}{3}(x^*)^{3/2}-t^*(d+1)/2=0, \qquad (17)$$

and

$$D(t^*)=[y(\alpha)-\alpha]/\alpha, \qquad (18)$$

where y is the root of

$$y^3+3y^2[1-\alpha-\tfrac{3}{4}]+3y(1-\alpha)^2+(1-\alpha)^3=0,$$

which tends to $\frac{9}{4}$ when $\alpha \to 1$. When α is large, y diverges like

$$y=\alpha+(3\alpha/2)^{2/3}+O(\alpha^{1/3}).$$

When $\alpha=1$, both expressions (9) and (18) reduce to the exact results (15). Therefore, the function $D(t^*)$ is given in the entire domain by

$$D(t^*)=\begin{cases}\phi(\alpha) & \text{if } \alpha\leqslant 1,\\ [y(\alpha)-\alpha]/\alpha & \text{if } \alpha\geqslant 1.\end{cases} \tag{19}$$

Therefore, we see that $D(t^*)$ is a function of a single variable, α, provided that $b \ll 1$. This function is displayed in Fig. 1. Since $\alpha(d)=\alpha(1/d)$, the result (19) for the reduced delay holds for good *and* bad cavities.

Of course when x_0 is negative, the condition of divergence (17) holds if and only if x_0 is not an exact zero of $\mathrm{Ai}(x)$ in which case $v(t)$ remains a bounded function of t forever.

The effect of delayed bifurcation can have interesting applications. In the case of Eqs. (1), we have verified that when we sweep backward ($b<0$) starting from the nontrivial steady solution $E^2=A-1$, the time-dependent solution closely follows the stationary solution and no significant delay occurs around $A=1$. Hence in a back and forth sweep there will be an hysteresis effect leading to dynamically induced optical bistability.

In real experiments, spontaneous emission is always present and therefore the trivial solution is never $E=0$, though the actual field amplitude is very small. Since our previous analysis exploited the property that $E=0$ remains an exact solution even when $A=A(t)$, we have to verify if a small departure from $E=0$ does affect our results or not. We assume that the effect of spontaneous emission can be described by adding a small perturbation to Eqs. (1), which we write in vector form as

$$\vec{z}_t=F(\vec{z},A(t))+\delta G(\vec{z},A(t)). \tag{20}$$

In this equation, δ measures the amplitude of the correction brought in by spontaneous emission or any other source of imperfection, and G is an $O(1)$ vector. When $\delta=0$, Eq. (20) reduces to (1).

When A is time independent and $\delta\rightarrow 0$, the perturbed steady branches $\vec{z}_s(A,\delta)$ approach the bifurcation branches $\vec{z}_s(A,0)$. By using the method of matched asymptotic expansions,[5] we can find the steady-state solutions. In the vicinity of $A=A_s=1$, they are approximated by

$$A=A_s+\delta^{1/2}A_1+O(\delta),$$

$$\vec{z}_s(A,\delta)=\delta^{1/3}\beta\vec{u}+O(\delta),$$

where $\vec{u}=(1,1,0)$ and the amplitude β is of order one in δ.

For a time-dependent pump parameter, we consider only small sweeping rates:

$$A(t)=A(0)+\epsilon t.$$

The following two extreme cases are easily dealt with: (i) When $0<\delta\ll\epsilon$ the results of our previous analysis remain correct except in a small vicinity of $A(t^*)$, provided that the initial condition satisfies

$$\vec{z}(0,\delta,\epsilon)=\epsilon\vec{z}_i=O(\epsilon).$$

(ii) When $0<\epsilon\ll\delta$ the time-dependent solution will follow the steady solution and no significant delay will occur. Therefore we see that a small imperfection added to the trivial solution may nevertheless preserve the delayed bifurcation for a class of sweeping rates and initial conditions.

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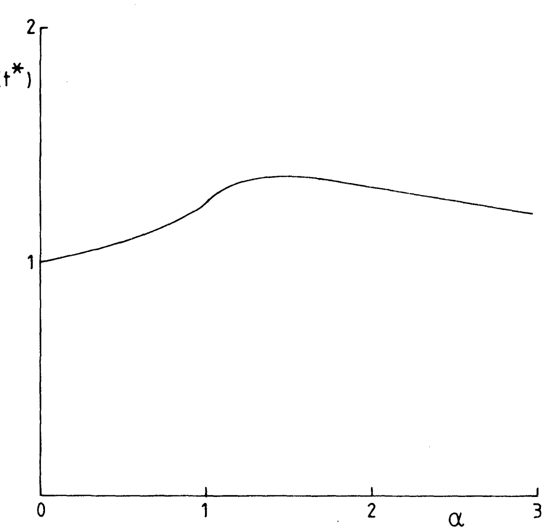

FIG. 1. Normalized delay vs α which is proportional to $-A(0)$.

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