Geometry, Topology, and Supersymmetry in Nonlinear Models

Thomas L. Curtright
Physics Department, University of Florida, Gainesville, Florida 32611

and

Cosmas K. Zachos
High Energy Physics Division, Argonne National Laboratory, Argonne, Illinois 60439
(Received 17 July 1984)

Supersymmetric two-dimensional sigma models which include Wess-Zumino topological terms are constructed, analyzed, and interpreted by means of torsion on the field manifold. One-loop renormalization results are presented geometrically, revealing that an infrared fixed point exists when the torsion parallelizes the manifold. The O(4)/O(3) model is used to illustrate general results.

PACS numbers: 11.10.Gh, 11.10.Lm, 11.30.Pb, 11.30.Rd

Multivalued actions of the Wess-Zumino type[1,2] are based on topological invariants in one higher space-time dimension and shift by an integral multiple of 2π under a symmetry transformation, thereby leaving the functional measure invariant. They have been studied in gauge theories,[3] but their impact on renormalization and their connection to anomalies is perhaps more easily explored in two-dimensional σ models.[4-6] Conventional two-dimensional σ models possess a rich geometry[7] which evolves systematically under renormalization.[8] In certain supersymmetric models,[9] however, this evolution is prevented by special geometric constraints and thus the associated β function vanishes identically.[10]

In this Letter, we investigate the geometrical significance of the Wess-Zumino term in two-dimensional σ models by constructing the supersymmetric extension of such bosonic models. This extension highlights the presence of torsion on the group manifold. For special finite values of the coupling, the torsion parallelizes the manifold,[11] i.e., the generalized curvature vanishes, and consequently the theory has an infrared fixed point. Witten[5] has previously noted the same β-function zeros in the pure bosonic model. We find that these zeros are also due to torsion parallelization. By way of illustration, we focus on the familiar O(4)/O(3) model, although more general results are given.

The conventional nonlinear O(4) model is obtainable from a pure kinetic term $(2\lambda^2)^{-1}\partial_\mu\phi^i\partial^\mu\phi^i$ $(i=0,1,2,3)$, by resolving the constraint in group space $(\phi^i\phi^i=1)$, to obtain

$$I_1=(2\lambda^2)^{-1}\int d^2x\, g_{ab}\partial^\mu\phi^a\partial_\mu\phi^b,$$
$$g_{ab}=\delta_{ab}+\frac{\phi^a\phi^b}{1-\phi^2}\quad (a,b=1,2,3). \tag{1}$$

This Lagrangian is manifestly invariant under linear vector O(3) (isospin) transformations, and is also invariant under three nonlinear axial transformations. In infinitesimal form we have

$$\delta\phi^a=\epsilon^{abc}\lambda^b_{\rm iso}\phi^c+\lambda^a_{\rm axi}(1-\phi^2)^{1/2}. \tag{2}$$

The pion fields ϕ^a are Goldstone bosons of the axial transformations, i.e., the projective coordinates of $O(4)/O(3)_{\rm isospin}$. In geometrical language, the above transformations correspond to isometries of the metric of the three-dimensional field manifold.

Similarly, resolving the constraint for the topological density in three space-time dimensions, $\epsilon^{\mu\nu\kappa}\epsilon^{ijkl}\phi^i\partial_\mu\phi^j\partial_\nu\phi^k\partial_\kappa\phi^l$, one obtains a total divergence. Through Gauss's law, this yields the Wess-Zumino term in two space-time dimensions:

$$I_2=(N/12\pi)\int d^2x\,\epsilon^{\mu\nu}e_{ab}\partial_\mu\phi^a\partial_\nu\phi^b,$$
$$e_{ab}=\epsilon_{abc}\phi^c f(\phi^2), \tag{3}$$
$$f(x)=\tfrac{3}{2}x^{-3/2}[\arcsin x^{1/2}-(x-x^2)^{1/2}],$$

where $f(0)=1$, and N is an integer—the Chern-Simons coefficient required by topology to make the action well-defined modulo $2\pi N$. I_2 is again manifestly isospin invariant, but under a general axial transformation it shifts by an integer multiple of 2π. For reasons to be explained, we shall call the antisymmetric tensor e_{ab} the "torsion potential."

It is easy to verify that the infinitesimal axial transformation is not an isometry of the torsion potential. Shifting ϕ by such an axial transformation induces a "gauge transformation" plus a standard general reparametrization transformation on the torsion potential:

$$\delta_{\rm gauge}e_{ab}=\partial_a\beta_b-\partial_b\beta_a,$$
$$\delta_{\rm std}e_{ab}=-\xi^c{}_{,a}e_{cb}-\xi^c{}_{,b}e_{ac}, \tag{4}$$

where $\xi^a = \lambda^a(1-\phi^2)^{1/2}$ and

$$\beta_a = \lambda^b \epsilon_{abc}\phi^c(f/2+\phi^2 f')(1-\phi^2)^{1/2}.$$

Consequently, the integrand in (3) transforms under (4) by a total divergence: $2\partial_\mu(\epsilon^{\mu\nu}\beta_a\partial_\nu\phi^a)$.

Further note that the curl of e_{ab} is itself a special curl-free and divergence-free tensor associated with the volume element (and group structure constants) of the manifold:

$$e_{[ab;c]} = e_{[ab,c]} = \epsilon_{abc}\sqrt{g}, \tag{5}$$

where $\sqrt{g} \equiv (\det g_{ab})^{1/2} = (1-\phi^2)^{-1/2}$, and $\Gamma^a{}_{(bc)} = \phi^a g_{bc}$. Since e_{ab} is divergenceless ($\delta e = 0$) and so is its curl ($\delta de = 0$), the Hodge-deRham operator[12] vanishes: $\Delta e = (d\delta + \delta d)e = 0$. Thus e_{ab} is a harmonic form.

Although I_1 is invariant under naive parity reversal $x \to -x$, as well as $\phi \to -\phi$, I_2 is only invariant under the combination of both these symmetries, as it has no terms with even numbers of pions.

The bosonic action in total is the sum of I_1 and I_2

$$I_B = (2\lambda^2)^{-1}\int d^2x [g_{ab}\partial^\mu\phi^a\partial_\mu\phi^b + \tfrac{2}{3}n\epsilon^{\mu\nu}e_{ab}\partial_\mu\phi^a\partial_\nu\phi^b], \tag{6}$$

where we have defined $n \equiv N\lambda^2/4\pi$ for future convenience. Written in this form,[13] the theory is straightforward to supersymmetrize by superfield extension of ϕ.

Given a Riemannian metric and an antisymmetric tensor e_{ab}, the following defines a supersymmetric model for general-field manifolds. However, certain conditions[5] are crucial for a topological interpretation:

$$\begin{gathered}\Phi^a = \phi^a + \bar\theta\psi^a + (\bar\theta\theta/2)F^a, \quad D\Phi^a = \psi^a + (F^a - i\partial\!\!\!/\phi^a)\theta + (\bar\theta\theta/2)i\partial\!\!\!/\psi^a, \\ I_s = (2\lambda^2)^{-1}\int d^2x(d^2\theta/2)[g(\Phi)_{ab} - \tfrac{2}{3}ne(\Phi)_{ab}]\bar D\Phi^a(1+\gamma_3)D\Phi^b.\end{gathered} \tag{7}$$

If θ is integrated out and the auxiliary fields are eliminated (relevant conventions and identities may be found elsewhere[14]), the supersymmetric Lagrangian reads

$$\begin{aligned}I_s = (2\lambda^2)^{-1}\int d^2x [&g_{ab}\partial_\mu\phi^a\partial^\mu\phi^b + ig_{ab}\bar\psi^a(\mathscr{D}\!\!\!/\psi)^b + \tfrac{2}{3}ne_{ab}\epsilon^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b \\ &- \tfrac{1}{3}n\partial_\mu(ie_{ab}\bar\psi^a\gamma_3\gamma^\mu\psi^b) + \tfrac{1}{8}\mathscr{R}_{abcd}\bar\psi^a(1+\gamma_3)\psi^c\bar\psi^b(1+\gamma_3)\psi^d].\end{aligned} \tag{8}$$

The antisymmetric tensor e_{ab} serves as the potential for the torsion S_{abc}, which is naturally incorporated into the covariant derivatives and the corresponding curvature:

$$\begin{gathered}S_{abc} = ne_{[ab;c]} = g_{ae}\Gamma^e{}_{[bc]}, \quad (\mathscr{D}_\mu\psi)^a \equiv \partial_\mu\psi^a + (\eta_{\mu\nu}\Gamma^a{}_{(bc)} + \epsilon_{\mu\nu}\Gamma^a{}_{[bc]})\partial^\nu\phi^b\psi^c, \\ \mathscr{R}_{abcd} \equiv R_{abcd} + (S_{aec}S^e{}_{bd} - S_{aed}S^e{}_{bc}) + g_{ae}(S^e{}_{bd;c} - S^e{}_{bc;d}).\end{gathered} \tag{9}$$

Here R_{abcd} is the conventional curvature constructed from the symmetric connection $\Gamma^a{}_{(bc)}$.

Specializing to the O(4)/O(3) case, whose topology was discussed earlier, we have $S_{abc} = ng^{1/2}\epsilon_{abc}$, hence $S_{abc;d} = 0$, and $\mathscr{R}_{abcd} = (1-n^2)R_{abcd}$. Therefore

$$\begin{aligned}I_s(O(4)/O(3)) = (2\lambda^2)^{-1}\int d^2x [&g_{ab}\partial_\mu\phi^a\partial^\mu\phi^b + ig_{ab}\bar\psi^a D\!\!\!\!/\,\psi^b + \tfrac{2}{3}n\epsilon^{\mu\nu}e_{ab}\partial_\mu\phi^a\partial_\nu\phi^b \\ &+ ing^{1/2}\epsilon_{abc}\partial_\mu\phi^a\bar\psi^b\gamma_3\gamma_\mu\psi^c + \tfrac{1}{6}(1-n^2)R_{abcd}\bar\psi^a\psi^c\bar\psi^b\psi^d],\end{aligned} \tag{10}$$

where $D_\mu\psi^a$ is the conventional[10] torsionless covariant derivative $\partial_\mu\psi^a + \Gamma^a{}_{(bc)}\partial_\mu\phi^b\psi^c$.

Now note that for special values of the coupling ($\lambda^2 = \pm 4\pi/N$, i.e., $n = \pm 1$) *the generalized curvature vanishes,* and therefore so do the four-fermion couplings: The manifold has been parallelized.[11] These couplings coincide with the zeros of the β function in the pure bosonic model,[5] which we now explain.

The renormalization of the geometry of the usual supersymmetric σ model is well studied.[10] By a similar analysis using the background-field method and Riemannian normal coordinates, we have computed the one-loop divergences of the model defined by I_s in (7), for general $g_{(ab)}$ and $e_{[ab]}$. We use the component form of the model as given in (8) in order to discuss both the bosonic σ model ($\psi \equiv 0$) and the supersymmetric case concurrently. Using dimensional regularization to evaluate momentum integrals in d space-time dimensions, we find that all one-loop on-shell ultraviolet divergences are removed by adding to the bare metric and torsion potentials the following counterterms:

$$\begin{gathered}g^{(1)}_{ab} = [\lambda^2/2\pi(2-d)]\mathscr{R}_{(ab)}, \\ \tfrac{2}{3}ne^{(1)}_{ab} = [\lambda^2/2\pi(2-d)]\mathscr{R}_{[ab]},\end{gathered} \tag{11}$$

which involve both the symmetric and antisymmetric parts of the generalized Ricci tensor with torsion

$$\mathscr{R}_{(ab)} \equiv R_{ab} + S^c{}_{da} S^d{}_{cb},$$
$$\mathscr{R}_{[ab]} \equiv S^c{}_{ab;c}. \qquad (12)$$

Note that in the "harmonic gauge," i.e., when $e^a{}_{b;a} = 0$, the second counterterm is $e^{(1)} = [\lambda^2/4\pi(2-d)]\Delta e$, where Δe is the Hodge-deRham operator acting on the torsion potential.

The presence or absence of the ψ terms in (8) has no effect on (11) and (12). The one-loop results are independent of the fermions in the model, so that (11) applies to either the pure bosonic or the supersymmetric version. This is a well-known feature of the usual σ model,[10] and is true here for essentially the same reason (fermions give a finite self-energy contribution to minimally coupled vectors in two dimensions).

If the Wess-Zumino term in (8) has a topological interpretation, we do not expect it to be renormalized. This is true (to one loop) if and only if $\mathscr{R}_{[ab]} = 0$, which means the torsion is a co-closed form.[12] We shall assume this in the following. It is always true for the O(4)/O(3) model that $\mathscr{R}_{[ab]} = 0$, even when $\mathscr{R}_{abcd}$ does not vanish [see the remarks after Eq. (5)]. This is also true for general chiral models. Indeed, we expect the topology for such general chiral models to be unaltered by radiative corrections to all orders.

However, in general we expect the geometry of the manifold to change under radiative corrections. In renormalization-group language,[8,10] a shift in mass scale alters the geometry to one loop by deforming the metric:

$$m(d/dM)(g_{ab}/\lambda^2) = (2\pi)^{-1}\mathscr{R}_{(ab)}. \qquad (13)$$

Accordingly, the geometry of the manifold encounters a fixed point when it is parallelized, $\mathscr{R}_{abcd} = 0$. For the O(4)/O(3) model, e.g., we have $\mathscr{R}_{ab} = 2(1-n^2)g_{ab}$, and thus there is an uv fixed point at $\lambda^2 = 0$ and a nontrivial ir fixed point at $\lambda^2 = |4\pi/N|$. This can be visualized in terms of the effective hypersphere in ϕ^i space which, for example, shrinks with decreasing energy down to a fixed critical radius.

Witten[5] has argued that the purely bosonic model is equivalent to a free-fermion theory at this infrared fixed point. Such an equivalence requires $M(d/dM)(g_{ab}/\lambda^2) = 0$ to *all* orders in perturbation theory. While this has not been proved, we note that parallelism of the manifold may be the crucial ingredient needed to establish the ir fixed point to all orders. It is not unreasonable to conjecture that the extension to higher orders of the right-hand side of (13), or (11), will be expressible in terms of products, traces, and covariant derivatives of the generalized curvature.

We thank P. G. O. Freund for discussions on "Fernparallelismus." We acknowledge financial support from the Department of Energy under Contract No. DE-AS-05-81ER4008.

Note added.—After submitting this paper for publication, we learned of an independent study by J. Gates, C. Hull, and M. Rocek [SUNY Stony Brook Report No. ITP-SB-84-53 (to be published)] in which torsion is also introduced and the possible symmetries of the classical theory defined by Eq. (7) are investigated, but without considering quantum effects. We thank J. Lukierski for calling our attention to this report.

[1]J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971).

[2]E. Witten, Nucl. Phys. **B223**, 422 (1983).

[3]W. Siegel, Nucl. Phys. **B156**, 135 (1979); J. Schonfeld, Nucl. Phys. **B185**, 157 (1981); S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982).

[4]A. Polyakov and P. Wiegmann, Phys. Lett. **131B**, 121 (1983), and **141B**, 223 (1984); P. Wiegmann, Phys. Lett. **141B**, 217 (1984).

[5]E. Witten, Commun. Math. Phys. **92**, 455 (1984).

[6]A. D'Adda, A. Davis, and P. DiVecchia, Phys. Lett. **121B**, 335 (1983); S. Novikov, Sov. Math. Dokl. **24**, 222 (1981); O. Alvarez, Nucl. Phys. **B238**, 61 (1984).

[7]K. Meetz, J. Math. Phys. **10**, 589 (1969).

[8]J. Honerkamp, Nucl. Phys. **B36**, 130 (1972); G. Ecker and J. Honerkamp, Nucl. Phys. **B35**, 481 (1971); W. Bardeen, B. Lee, and R. Shrock, Phys. Rev. D **14**, 985 (1976); D. Friedan, Phys. Rev. Lett. **45**, 1057 (1980).

[9]T. Curtright and D. Z. Freedman, Phys. Lett. **90B**, 71 (1980); L. Alvarez-Gaumé and D. Z. Freedman, Phys. Lett. **94B**, 171 (1980).

[10]L. Alvarez-Gaume, D. Z. Freedman, and S. Mukhi, Ann. Phys. N. Y. **134**, 85 (1981); L. Alvarez-Gaume and D. Z. Freedman, Commun. Math. Phys. **80**, 443 (1981).

[11]E. Cartan and J. Schouten, Proc. K. Ned. Akad. Wet. Amsterdam, **29**, 803,933 (1926); S. Goldberg, *Curvature and Homology* (Academic, New York, 1962); also see A. Einstein, Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl. **1928**, 217.

[12]T. Eguchi, P. Gilkey, and A. Hanson, Phys. Rep. **66**, 213 (1980).

[13]We remind the reader that the simplicity of (2) dictates a normalization for the pion which complicates the manifestly chiral exponential notation. Thus, in our choice of coordinates the chiral matrix is $U = \exp[i\vec{\tau}\cdot\vec{\phi}h(\phi^2)]$, where $h(x) = x^{-1/2}\operatorname{arcot}[(1-x)/x]^{1/2}$.

[14]T. Uematsu and C. Zachos, Nucl. Phys. B **201**, 250 (1982).