Partition Function of the Three-Dimensional Zamolodchikov Model

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It is shown that the Zamolodchikov model on the cubic lattice of N sites is equivalent to a spin model on a lattice of $2N$ sites with just two-spin and three-spin interactions. Together with the known transfer-matrix commutativity, this implies that the partition function Z (for N large) factors into a product of single-variable functions. These functions, and hence Z, are calculated by using symmetry relations. The model appears to be a critical free-fermion model.

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Zamolodchikov[1,2] proposed a solution of the tetrahedron equations. From the point of view of statistical mechanics these equations are the conditions for two layer-to-layer transfer matrices to commute. The author[3] expressed Zamolodchikov's solution as a spin model on the cubic lattice, and verified that the tetrahedron equations are indeed satisfied.

The Boltzmann weights of the model are real, but some are negative. Despite this unphysical property, it is still interesting to calculate the partition function since this is presently the only three-dimensional model that appears to be both nontrivial and solvable. Here this is done by using the commutation and factorization properties of the transfer matrix together with symmetry properties: The calculation is rather like the "399th" solution of the Ising model.[4,5]

Let $\mathscr{L}$ be the simple-cubic lattice with m layers each of n sites, and $N=mn$ sites in all. Impose cyclic boundary conditions and with each site i associate a spin σ_i, with values ± 1. The partition function is

$$Z=\sum_{\{\sigma\}}\prod_{\text{cubes}} W(a|efg|bcd|h), \tag{1}$$

where the product is over all elementary cubes; $a,\ldots,h$ are the eight corner spins of an individual cube, arranged as in Fig. 1, and $W(a|efg|bcd|h)$ is the Boltzmann weight of the cube; the sum is over all values of the N spins.

The values of W are given in Table I and Eqs. (3.11)–(3.13) of Ref. 3. They depend on three angles ϕ_1, ϕ_2, and ϕ_3, herein called θ_1, θ_2, and θ_3, and the associated quantities ($i=1,2,3$)

$$2\alpha_0=\theta_1+\theta_2+\theta_3-\pi,\quad \alpha_i=\theta_i-\alpha_0. \tag{2}$$

It is natural to regard θ_1, θ_2, and θ_3 as the angles of a spherical triangle. Then the $2\alpha_i$ are the spherical excesses, and the sides are a_1, a_2, and a_3, where[6]

$$\sin\theta_1\sin\theta_2\cos a_3=\cos\theta_3+\cos\theta_1\cos\theta_2 \tag{3}$$

(similarly for a_1 and a_2). The aim here is to calculate the partition function per site

$$\kappa=Z^{1/N} \tag{4}$$

in the large-lattice limit, when the θ_i, α_i, and a_i are all real, nonnegative, and less than π.

Let Y_{abfg} by any function of the four spins, a, f, b, and g. Then the product in (1) is unchanged by multiplying $W(a\ldots h)$ by

$$\frac{Y_{edhc}Y_{gcea}Y_{chbg}}{Y_{afbg}Y_{bhdf}Y_{edfa}}, \tag{5}$$

since each face of $\mathscr{L}$ acquires a Y function from one of its adjoining cubes, and a canceling $1/Y$ from the other cube. Similarly, (1) is unchanged by multiplying W by

$$\exp[i\pi(af-ch+dh-ag+cg-df)/8]. \tag{6}$$

The effect of doing both, with

$$Y_{afbg}=\exp[i\pi(ab+f)(g-f)/8],$$

is to multiply the weights given in rows 3, 6, 7, and 8 of Table I of Ref. 3 by icd, ac, iac, and cd, respectively. The new function $\overline{W}$ is unchanged by negating the top four spins a, f, b, and g (or the four bottom spins e, d, h, and c). Also, when $\alpha_0=\alpha_2=0$,

$$\overline{W}(a|efg|bcd|h)=\delta_{af,ch}, \tag{7}$$

so that from (1) and (4) it then follows that $\kappa=1$.

Let $\phi_{a,b}=-1$ if $a=b=-1$, otherwise $\phi_{a,b}=+1$. Write t_i and c_i for $[\tan(\alpha_i/2)]^{1/2}$ and $[\cos(\alpha_i/2)]^{1/2}$, respectively. Define

$$\gamma=(c_0c_1c_2c_3)^{-1}, \tag{8}$$

$$\xi=\tfrac{1}{2}\gamma(\tfrac{1}{2}\sin\theta_3)^{1/2}, \tag{9}$$

$$\begin{aligned}&\tanh x=t_0t_3,\quad \tan y=t_2/t_1,\\ &\tanh x'=t_1t_2,\quad \tan y'=t_0/t_3,\\ &2K_1=-x'-iy',\quad 2K_2=x-iy,\\ &2K_3=-x'+iy',\quad 2K_4=x+iy.\end{aligned} \tag{10}$$

The key step in this working was the discovery that

$$\overline{W} = \xi e^{iD/8} \sum_s \phi_{s,af} \phi_{s,ch} \exp[s(K_1 ag + K_2 bf + K_3 dh + K_4 ce)], \tag{11}$$

where s takes the values ± 1, and

$$D = 2\pi(af - ch) + (\pi - \theta_1)(cdeh - abfg) + \theta_2(aecg - bhdf) + \theta_3(bhcg - aedf). \tag{12}$$

Just as the factors (5) and (6) cancel out of (1), so does the factor $\exp(iD/8)$ in (11), and so we can ignore it. We can regard s as a spin located at the center of the cube, as in Fig. 1. From (1) and (11) it follows that $\xi^{-N}Z$ is the partition function of a bcc lattice of $2N$ sites, consisting of L and the body centers of each cube. There are three-spin interactions $s_i \sigma_j \sigma_k$ on the shaded triangles in Fig. 1, with interaction coefficients $K_1, \ldots, K_4$. Since $\phi_{s,af} = \phi_{s,a}\phi_{s,f}$, there are sign factors $\phi_{s,a}\phi_{s,f}\phi_{s,c}\phi_{s,h}$ associated with the edges denoted by heavy lines in Fig. 1.

The coefficients $K_1, \ldots, K_4$ are not independent: If we set $z = \exp(i\alpha_3/2)$, $v_i = \tanh 2K_i$, and

$$T_i = [\tan(\theta_i/2)]^{1/2}, \tag{13}$$

it follows from spherical trigonometry[6] that

$$v_1 = -zT_1T_2, \quad v_2 = -izT_2/T_1,$$

$$v_3 = -z^{-1}T_1T_2, \quad v_4 = iz^{-1}T_2/T_1,$$

$$v_1v_4 + v_2v_3 = 0. \tag{14}$$

Since the σ spins in each layer interact only with the s spins immediately above and below them (and vice versa), the transfer matrix T can be factored:

$$T = \xi^n X(K_3,K_4)\, Y(K_1,K_2), \tag{15}$$

where X depends on θ_1, θ_2, and θ_3 only via the interaction coefficients K_3 and K_4 of the lower shaded triangles in Fig. 1; Y depends only on K_1 and K_2.

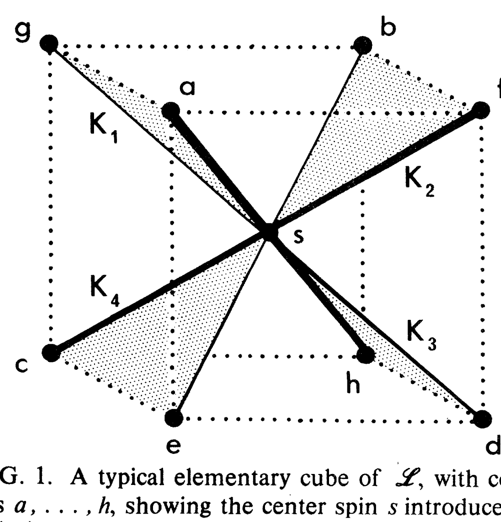

FIG. 1. A typical elementary cube of $\mathscr{L}$, with corner spins $a, \ldots, h$, showing the center spin s introduced via Eq. (11).

For the moment regard θ_1, and hence T_1, as fixed. Then $v_4 = -iv_3/T_1^2$, so that X depends on θ_2 and θ_3 only via v_3. Similarly, Y depends on them only via v_1.

In Refs. 2 and 3 it was shown that the transfer matrices of two models commute provided that they have the same value of θ_1. This property is unchanged by the face and edge transformations involved in replacing W by $\overline{W}$ and neglecting D in (11). If we write X as $X(v_3)$ and Y as $Y(v_1)$, it follows that $X(u)Y(v)$ commutes with $X(u')Y(v')$, for all complex numbers u, v, u', and v'. Under quite general conditions (in particular, provided that the eigenvectors of T span its vector space), it follows that there exist nonsingular matrices P, Q, and diagonal matrices $R(u)$ and $S(v)$, such that

$$X(u) = PR(u)Q^{-1}, \quad Y(v) = QS(v)P^{-1}, \tag{16}$$

where P and Q are independent of u and v.

The relation $Z = \operatorname{Tr} T^m$ gives

$$Z = \xi^N \operatorname{Tr}[R(v_3)S(v_1)]^m. \tag{17}$$

The diagonal matrix $R(v_3)$ depends on θ_1 as well as v_3: We can exhibit this by writing it as $R(v_3, v_4)$. Similarly, we can write S as $S(v_1, v_2)$. Suppose that T has a unique largest-modulus eigenvalue and let $r^n(v_3, v_4)$ and $s^n(v_1, v_2)$ be the corresponding diagonal elements of R and S. Then, taking the limit of m large and using (4), we obtain

$$\kappa = \xi r(v_3, v_4) s(v_1, v_2). \tag{18}$$

Thus κ/ξ, which is a function of the three angles θ_1, θ_2, and θ_3, factors into a product of functions of only two variables. This property, together with some simple symmetries of κ, determines r, s, and κ.

From Ref. 3 it is readily seen that permuting θ_1, θ_2, and θ_3 is merely equivalent to rotating the lattice $\mathscr{L}$, and so leaves κ unchanged. More strongly, κ is unchanged by permuting α_0, α_1, α_2, and α_3. The interchanging of θ_1 and θ_2 in (18), using (15)–(17), gives

$$\kappa = \xi r(v_3, v_2^{-1}) s(v_1, v_4^{-1}). \tag{19}$$

Eliminating κ between (18) and (19) gives an identity which must be true for all complex numbers v_1, v_2, v_3, and v_4 (in appropriate domains) satisfying (14). By taking logarithms and differentiating while keeping v_3 and v_4 fixed, we can establish that there exist single-variable functions f, g, h, and p such that

$$r(u,v)=g(u)h(v)p(u/v),$$
$$s(u,v)=f(u)h(v^{-1})/p(-u/v), \quad (20)$$
$$\kappa=\xi f(v_1)h(v_2^{-1})g(v_3)h(v_4).$$

The $\theta_1 \leftrightarrow \theta_3$ interchange symmetry can now be used to determine the functions f, g, and h to within a few unknown constants. [The working is technical and will be given in a later paper: It helps to consider the case when the lengths a_1, a_2, and a_3 and the area α_0 are small—this is Zamolodchikov's "static limit"—and to expand (20) to first order in the length scale.]

The constants can then be determined by using the $\alpha_0, \ldots, \alpha_3$ permutation symmetry and the requirement that $\kappa=1$ when $\alpha_0=\alpha_2=0$. The results for f, g, and h are (to within multiplicative constants)

$$f(v)=(1-v^2)^{-1/4}F_-(v^2),$$
$$g(v)=(1-v^2)^{-1/4}/F_-(v^2), \quad (21)$$
$$h(v)=F_+(v^2),$$

where $F_\pm(x)=\exp[iG_\pm(x)/(4\pi)]$ and

$$G_\pm(x)=\int^x\left(\frac{\ln(1\pm y)}{y}-\frac{\ln y}{y\pm 1}\right)dy. \quad (22)$$

Finally, κ is given by

$$\ln\kappa=\ln\gamma-\phi(s)+L_1+L_2+L_3, \quad (23)$$

where γ is given by (8), $s=(a_1+a_2+a_3)/2$,

$$\phi(x)=\sum_{n=1}^{\infty}(\sin 2nx)/2\pi n^2, \quad L_i=\phi(s-a_i)+\{a_i\ln[2\pi n^2\sin(\theta_i/2)]+(\pi-a_i)\ln\cos(\theta_i/2)\}/(2\pi).$$

This expression for κ has the required property that it be unchanged by replacing the spherical triangle by any of its colunar triangles, i.e., by permuting α_0, α_1, α_2, and α_3. It also satisfies the "unitarity condition" or "inverse relation," Eq. (5.2) of Ref. 2 (Z therein is our κ; $Z^{(\alpha)}$ is κ with all the lengths of the spherical triangle negated). This relation has been known for three years, but κ has not been derived from it because no information was available on the analytic properties of κ. This contrasts with two dimensions (where the inversion relation plus simple analyticity assumptions provide a quick way of calculating κ),[7-10] and highlights the importance of the analyticity properties in the inversion-relation method.

In the isotropic case, where $2\alpha_0=2\alpha_i=\theta_i=a_i=\pi/2$ (for $i=1,2,3$), (23) gives

$$\kappa=2^{3/4}(2^{1/2}-1)e^{2G/\pi}=1.2480\ldots,$$

where $G=0.915\,965\ldots$ is Catalan's constant. This fits well with recent numerical estimates.[11]

The occurrence of the integrals $G_\pm(x)$, and of Catalan's constant, is strongly reminiscent of the two-dimensional critical Ising and free-fermion models. Indeed, if $\mathscr{L}$ is only two layers thick, the Zamolodchikov model becomes a planar critical free-fermion model. Recent numerical work[11] on a more general model (one for which low-temperature expansions can be obtained) suggests that its spontaneous magnetization vanishes at the particular temperature corresponding to the Zamolodchikov model. Altogether, it seems likely that the Zamolodchikov model (for L infinitely thick) is critical.

It also seems likely, as has been suggested by Zamolodchikov[12] and Foerster,[13] that the model is in some sense a free-fermion model. Indeed, the form (11) of $\overline{W}$ was discovered by considering the plane through a, b, e, and h in Fig. 1, and noting that for any fixed values of c, d, f, and g the function $\overline{W}$ was that of a planar free-fermion model[14] and hence of a checkerboard Ising model.[15,16]

Let τ_r denote the product of the four spins around a face r of $\mathscr{L}$. Apart from sign factors the weight function W is a function only of the six τ spins on the faces of the cube in Fig. 1. Further, the overall sign of the product in (1) depends only on the $3N\ \tau$ spins in $\mathscr{L}$ (because it is unchanged by negating all the σ spins in any horizontal or vertical plane). It follows that we can express the Zamolodchikov model as a vertex model by using the spin $\rightarrow$ arrow transformation of Pearce and Baxter,[17] giving

$$Z=\sum_P(\pm)\Phi_P,$$

where the sum is over allowed arrow configurations, Φ_P is the product of vertex weights, and the sign has to be calculated globally. This looks a little like Bazhanov and Stroganov's "sum over polygons" formulation of the three-dimensional free-fermion model.[18]

The s spins introduced via (11) are on a different footing from the original σ spins. A transformation that removes this asymmetry is to locate a spin p (q) at the center of the front (back) face in Fig. 1, and to set $s=pq$. Do this for all elementary cubes of $\mathscr{L}$, identifying spins in the same position. (Thus p is also associated with the cube in front of it: This changes the boundary conditions, but is otherwise permissible.) If one notes that $\phi_{pq,a}=\phi_{p,a}\phi_{q,a}$, and rotates the lattice, $\xi^{-N}Z$ becomes the partition function of a spin model on a cubic lattice of $2N$ sites, with pure four-spin interactions on vertical faces only, $\phi_{\sigma,\sigma'}$ factors on all horizontal edges, and in each vertical face a $\phi_{\sigma,\sigma'}$ factor on one diagonal (e.g., top right to lower left) only. The sign factors $\phi_{\sigma,\sigma'}$ are important: Without them the model would trivially factor into independent planar checkerboard Ising models, being simply a checkerboard generalization of Suzuki's model.[19]

There are many ways of writing (23): The author is indebted to Professor M. L. Glasser for helping to obtain what appears to be the simplest form.

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