Eden Model in Many Dimensions

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(Received 25 May 1984)

The Eden model in many dimensions is studied by an exact enumeration method. Our result including the first-order $1/d$ correction has the asymptotic behavior $\langle R_n^2\rangle \sim 2\ln(n)(1+3/2d)$, which does not agree with the naive expectation $R \sim n^{1/d}$ [or $d(n^{1/d}-1)$]. This suggests that when n is large, $\langle R_n^2\rangle$ might be a singular function of d. We also present a result on the Cayley tree.

PACS numbers: 02.50.+s, 05.70.Ln, 68.70.+w

Recently there has been a great deal of interest in the kinetic cluster growth problem, in particular in the diffusion-limited aggregation (DLA) model proposed by Witten and Sander.[1] Most of the work on this problem is based on computer simulations[1,2] and on the study of "mean-field"-type equations.[3,4] There has also been an attempt to use real-space renormalization-group techniques.[5]

The Eden model[6] is perhaps the simplest one which describes a growing cluster: A particle on the perimeter of the cluster acquires a new particle on its empty nearest-neighbor sites with equal probability (properly normalized). In the DLA model there are screening effects which differentiate the exposed perimeter particles from the shadowed ones by assigning them different probabilities. However, in infinite dimensions one would expect the excluded-volume effects (here the screening effects) to disappear and thus all particles would become equally exposed. The DLA model and the Eden model should coincide in infinite dimensions (see also Vannimenus, Nickel, and Hakim[7]). The study of the Eden model in many but finite dimensions may shed light on the corresponding DLA problem. (The connection between the Eden model in finite dimensions and DLA has been discussed recently by Plischke and Ràcz.[8])

One might expect the growing clusters in the Eden model to be "compact" and the mean square size (to be defined later) of a cluster to have a quite simple asymptotic behavior $\langle R_n^2\rangle \underset{n\to\infty}{\sim} n^{1/d}$ or

$$\langle R_n^2\rangle \underset{n\to\infty}{\sim} d(n^{1/d}-1) \tag{1}$$

as the Monte Carlo study[9] has suggested for $d=2,3$. Throughout this Letter d is the Euclidean dimensionality.

If $\langle R_n^2\rangle$ is a smooth function of d for large n then one should be able to obtain the following expansion in $1/d$ using Eq. (1):

$$\langle R_n^2\rangle \underset{n\to\infty}{\sim} \ln(n)+[\ln(n)]^2/d+O(1/d^2).$$

In this Letter, we have studied the Eden model in many dimensions. Using the exact enumeration and recursion relations we obtained the surprising result

$$\langle R_n^2\rangle \underset{n\to\infty}{\sim} 2\ln(n)(1+3/2d)$$

which does not agree with Eq. (1). Our result implies that the limits $d\to\infty$ and $n\to\infty$ cannot be naively interchanged, that Eq. (1) cannot hold for all d, and that there might exist a critical value d_c above which $R_n^2 \sim \ln(n)$. However, to obtain a *conclusive* answer one should study higher orders in the $1/d$ expansion which is very tedious in our approach. In view of our present result, we would like to draw attention to the possibility of critical large-d behavior in the Eden and DLA models.

In the following, we briefly explain our enumeration recursion method first for $d=\infty$ and then include the $1/d$ correction. To have a definite idea of the Eden model in infinite dimensions, which is equivalent to keeping the leading order in the $1/d$ expansion, we write down the first few order diagrams in Fig. 1. The diagrams only indicate the topology of the clusters. In many dimensions, the probability of two bonds being parallel is proportional to $1/d$, so that in the computation we can consider all bonds to be orthogonal to each other. The numbers in front of the diagrams are the relative weights, which are the probabilities for these diagrams to occur when multiplied by the normalization constant $1/(n-1)!$. For simplicity we have suppressed a factor $(2d)^n$ which comes from the number of ways in which the diagrams can be embedded on the lattice. The normalization can be also easily understood: It is just the sum of the to-

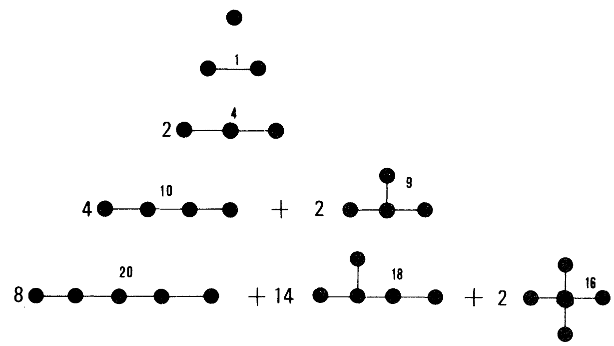

FIG. 1. The first five orders of diagrams of the Eden model in infinite dimension. See the text for details.

tal weights for order n.

For a single diagram we can define a mean square radius

$$l_g = \sum_{i<j} (x_i - x_j)^2 = \sum_{i<j} l_{ij}, \tag{2}$$

where the sum runs over all the sites of the diagram under consideration and l_{ij} is the distance from site i to site j measured along the bonds of the diagram. The second equality follows from the fact that all bonds are orthogonal to each other. The first few l_g's are also indicated by the numbers over the diagrams in Fig. 1. The normalization factor for l_g is $2/n(n-1)$, which is one over the total number of terms in the sum (2).

Next we would like to have an ensemble-averaged square size (i.e., mean square size), which is defined by

$$\langle R_n^2 \rangle = \frac{2}{n!(n-1)} S_n, \tag{3}$$

$$S_n = \sum\nolimits_g w_g l_g \tag{4}$$

where w_g is the weight for a diagram and the sum runs over all the topologically distinct diagrams for order n.

We have found the following recursion relation for the integer-valued function S_n:

$$S_{n+1} = (n+2) S_n + n \times n!. \tag{5}$$

We present only the main idea used to obtain Eq. (5), omitting the details, which are quite tedious. We take two successive orders for which we can calculate S_n and S_{n+1} by hand using Eq. (4) (see also Fig. 1). The contribution to S_{n+1} from the diagrams of order $n+1$ can be divided into two parts: One part is the contribution from the old diagrams which is proportional to S_n, and the other part is the contribution in which the new particle is involved. From this analysis we can write down a recursion relation for generic n and then check it for a few higher orders. It is in this way that we have derived Eq. (5).

From (3) and (5) with some algebra we can easily find the asymptotic behavior:

$$\langle R_n^2 \rangle \underset{n \to \infty}{\sim} 2\ln(n) - 2(2-C) + 4\frac{\ln(n)}{n} + \frac{5 - 4(2-C)}{n}, \tag{6}$$

where C is the Euler constant.

We may also be interested in higher moments. In place of Eq. (2), for the kth moment we now have

$$l_g^{(k)} = \sum_{i<j} (x_i - x_j)^{2k} = \sum_{i<j} l_{ij}^k, \tag{7}$$

and the ensemble average is

$$\langle R_n^{2k} \rangle = \frac{2}{n!(n-1)} S_n^{(k)}, \tag{8}$$

$$S_n^{(k)} = \sum\nolimits_g w_g l_g^{(k)}. \tag{9}$$

We have also found the recursion relation (details omitted):

$$S_{n+1}^{(k)} = (n+2) S_n^{(k)} + n \times n! + 2 \sum_{l=1}^{k-1} \begin{pmatrix} k \\ l \end{pmatrix} S_k^{(l)}, \tag{10}$$

where $S_n^{(1)} = S_n$.

In principle, we can solve Eq. (10) consecutively for all higher-order moments. The second moment is of particular interest, the corresponding recursion relation being

$$S_{n+1}^{(2)} = (n+2) S_n^{(2)} + n \times n! + 4 S_n. \tag{11}$$

Subsituting the solutions of (11) into (8) gives us $\langle R_n^4 \rangle$ in closed form, which enables us to calculate the fluctuations around the mean square size. The leading contribution that we found is

$$\delta^2 = \frac{[\langle R_n^4 \rangle - (\langle R_n^2 \rangle)^2]}{(\langle R_n^2 \rangle)^2} \sim \frac{1}{2\ln(n)}, \tag{12}$$

which implies that the fluctuations still exist in infinite dimensions and that they are small only on a logarithmic scale. This information may be useful for the mean-field approximations to the cluster growth models.

Next we would like to calculate the $1/d$ corrections, which are important if the many-dimension analysis is to make contact with reality. To get the first-order corrections, we do not have to worry about closed diagrams since they are suppressed by a factor of at least $(1/d)^3$ for a hypercubic lattice. Here we are concerned with the correction to $\langle R_n^2\rangle$.

In Fig. 2 we list the diagrams of the first few orders. S_n corresponds to the contribution from the old diagrams listed in Fig. 1. The normalization factor is almost the same as before, but there is an extra normalization due to the corrections and it is written explicitly in front. We define T_n as the coefficient of $-1/2d$.

By careful but tedious geometrical and algebraic analysis examining how T_{n+1} and T_n are related, similar to the analysis leading to (5), we arrive at the following recursion relation (with details again omitted):

$$T_{n+1}=(n+2)T_n+n^2\Big[2(n-2)(n-2)!+(n-1)!\sum_{k=1}^{n-2}2(k-1)/k\Big]$$
$$+2nS_n+4[S_{n-1}+(n-1)S_{n-2}+\ldots+(n-1)(n-2)\cdots 3S_2]. \tag{13}$$

From this relation we should be able to find T_n in closed form since S_n is known. Then we can compute the mean square size with the first-order $1/d$ correction by the following formula, which is symbolically expressed in Fig. 2:

$$\langle R_n^2\rangle=\frac{2}{n!(n-1)}\left(1+\frac{1}{2d}\sum_{k=1}^{n-1}\frac{2(k-1)}{k}\right)\left(S_n-\frac{T_n}{2d}\right), \tag{14}$$

which has the asymptotic behavior $\langle R_n^2\rangle\sim 2\ln(n)\times(1+3/2d)$. This is the rather unexpected result that we stated at the beginning of this Letter.

We feel that it is possible but exceedingly complicated to compute still higher corrections in $1/d$. Perhaps one should try to find only the asymptotic behavior for large n instead of attempting tedious exact enumerations.

Equation (13) is obtained by many intricate steps and it is very crucial for our final result. Is there any crosscheck on whether it is correct? For this reason we have performed direct countings of S_n and T_n from diagrams in Figs. 1 and 2, for $n\leq 8$ (we display the diagrams for n up to 5 only). We list these numbers in Table I and we have checked that (5) and (13) hold (we invite the readers also to do the same).

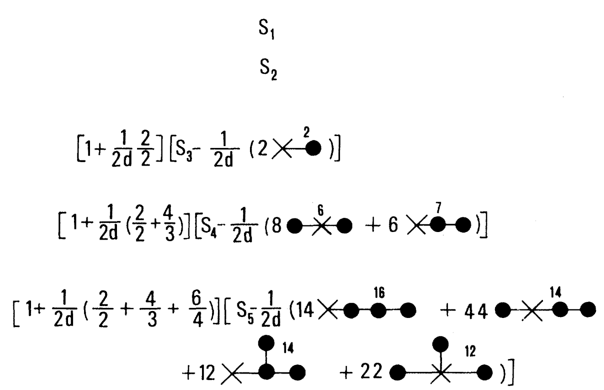

FIG. 2. The Eden model in many dimensions. Here we have taken into account the excluded-volume effects to the level $1/d$. The factors in front record modifications to the normalizations in Fig. 1. S_n denotes the corresponding part in Fig. 1. A cross implies two particles at the same point. (Note that two particles never occupy the same point in the Eden model; however, we have arranged in the computational process for such diagrams to occur in subtracted parts for simplicity.)

Might there still be a logical error involved, for example, in producing the relative weights in Fig. 2? To ensure that the overall scheme is correct, we have done a mini Monte Carlo simulation in which only a few particles are introduced (typically $n=8$) for various dimensions ($d=2,\ldots,20$). The program follows the exact rules of the Eden model and ignores all tricks that we have used in analytic computations. From the results of this simulation we have isolated the leading and next to leading contributions to check the exact formula (14) numerically for $n\leq 8$. Equation (14) is confirmed (this small system allows very good statistics, the typical iteration number being 10^5 and the statistical error about 3% or less, with 1 h of Vax/780 CPU time).

It also seems interesting to study the Eden model on a Cayley tree, which excludes any closed configuration. We have solved the Eden model on an arbitrary q-coordination Cayley tree in a way similar

TABLE I. The first few orders of S_n and T_n ($n\leq 8$).

n	S_n	T_n
2	1	0
3	8	4
4	58	90
5	444	1272
6	3708	16 052
7	33 984	200 200
8	341 136	2565 792

to that described before and we present only the result below. We have the following recursion relation defining the auxiliary quantity $C(q,n)$ for each value of q:

$$C(q,n+1)=C(q,n)+(q-2)^{-2}\{f(q,n)-\sum_{k=2}^{n-1}h(q,n,k)C(q,k)\},$$
$$f(q,n)=n(q-1)\Gamma\left(1+\frac{2q}{q-2}\right)\Gamma\left(n+\frac{q}{q-2}\right)\left[q\Gamma\left(1+\frac{2(q-1)}{q-2}\right)\Gamma\left(n+\frac{3q-2}{q-2}\right)\right]^{-1}, \tag{15}$$
$$h(q,n,k)=4(q-1)\Gamma\left(k+\frac{2q}{q-2}\right)\Gamma(n)\left[\Gamma\left(n+\frac{3q-2}{q-2}\right)\Gamma(k+1)\right]^{-1}.$$

From the above three formulas we can obtain $C(q,n)$ in closed form (in principle, of course).

The relation of $C(q,n)$ to the mean size (instead of mean square size) with the proper normalization is given by $\langle R(q,n)\rangle=[2(q-2)^2/nf(q,n-1)]\times C(q,n)$. Although the formula is complicated for an arbitrary pair (q,n), the leading large-n behavior if simple (although not easy to derive):

$$\langle R(q,n)\rangle \underset{n\to\infty}{\sim} 2\left(1+\frac{1}{q-2}\right)\ln(n) \tag{16}$$

In a recent work[7] by Vannimenus, Nickel, and Hakim, the Eden model on the Cayley tree is solved and the relationships among the Eden model on the Cayley tree and in infinite dimensions and the DLA model are discussed. They use a different and perhaps more mathematically rigorous technique. Our result, Eq. (16), on the Cayley tree seems to agree with their gyration radius [Eq. (28)] up to a multiplicative constant. The Cayley tree has a peculiar topology and we do not expect that any connection exists between the Cayley tree and other lattices.

To summarize, in this Letter we have studied the Eden model in many dimensions. The result in infinite dimensions, $\langle R_n^2\rangle \sim \ln(n)$, perhaps is not too surprising. If $\langle R_n^2\rangle$ as a function of d is smooth, then the naive expectation would be $\langle R_n^2\rangle \sim d(n^{1/d}-1)$, which is consistent with Monte Carlo results in two and three dimensions. Our result including the first $1/d$ correction $\langle R_n^2\rangle \sim \ln(n)(1+3/2d)$, reveals that there are surprises for the Eden mdoel for large values of d, and perhaps for other cluster growth models such as the DLA model as well.

Intuitively we would like to understand why the Eden model can have critical dimensionality. In sufficiently low dimensions, one would expect the Eden model to have Euclidean dimensionality[1]; however, as the dimensionality goes up the surface portions become more and more important, and thus the fluctuations become important. There might be a critical d_c above which the Euclidean dimensionality rule no longer applies to the Eden model.

The first-order correction is not sufficient to determine definitively whether or not a singularity exists nor what kind of singularity might be expected if one does exist, but it does suggest such a possibility. We hope that our study may stimulate other work on these ideas.

We thank L. Kadanoff and L. Peliti for helpful discussions.

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