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Coupling-Constant Metamorphosis and Duality between Integrable Hamiltonian Systems

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We introduce a noncanonical ("new-time") transformation which exchanges the roles of a coupling constant and the energy in Hamiltonian systems while preserving integrability. In this way we can construct new integrable systems and, for example, explain the observed duality between the Hénon-Heiles and Holt models. It is shown that the transformation can sometimes connect weak- and full-Painlevé Hamiltonians. We also discuss quantum integrability and find the origin of the deformation $-\frac{5}{72}\hbar^2 x^{-2}$.

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The search for (and the discovery of) integrable dynamical systems is a most fascinating branch of nonlinear physics, one which has been the center of intensive activity in the past decade.[1] Integrable systems are quite rare and still only a few examples are known. In this paper we will present a novel transformation that relates integrable Hamiltonian systems.

Several methods have been devised for the investigation of integrability. One method that has met with particular success in the last few years is singularity analysis, which associates integrability with the Painlevé property, i.e., a movable polelike singularity $(t-t_0)^{-n}$ in the solution of the equations of motion. It was used a century ago by Kowalevskaya,[2] who identified with it the last integrable configuration of the heavy top. The method was resurrected by Ablowitz, Ramani, and Segur[3] and by now several works which have combined Painlevé analysis with explicit construction of constants of motion have yielded new integrable systems (see, e.g., Chang, Tabor, and Weiss, Bountis, Segur, and Vivaldi, and Menyuk, Chen, and Lee[4]). However, it was soon found that there are integrable models which do not satisfy the above "full Painlevé" property. This led to the introduction of the "weak Painlevé" concept,[5,6] i.e., expansions involving movable branch points of the type $(t-t_0)^{1/r}$ (r integer). We will discuss below the relationship between these two types of Painlevé expansions.

Once an integrable potential is obtained a most interesting question can be asked: What are the possible perturbations of the potential which do not destroy integrability? Such additive terms are important, because whenever such terms with a free coupling constant are found they can lead to new integrable systems.[7]

In this paper the additive term is used in conjunction with a noncanonical ("new-time") transformation, which exchanges the roles of the coupling constant and the energy while preserving integrability. This transformation becomes even more powerful when a few judiciously chosen canonical transformations are applied in addition. For example we can now give a direct proof of the duality between the Hénon-Heiles and Holt systems. (By duality we mean that the integrability of one implies the integrability of the other.) This duality was observed by one of us[8] using Painlevé analysis, but at that time the underlying mechanism remained obscure. There are two further features that this transformation has with respect to (i) the Painlevé property and (ii) quantum integrability:

(i) Elsewhere[6] the singularity structure of the Holt and Fokas-Lagerstrom Hamiltonians was examined and found to be of the weak-Painlevé type. In what follows we will show that through this new transformation the above weak-Painlevé systems can be converted to full Painlevé.

(ii) We have found previously that for quantum integrability the Holt[8–10] and Fokas[9,10] Hamiltonians must be deformed by the surprising additional terms, $-\frac{5}{72}\hbar^2 x^{-2}$ and $-\frac{5}{72}\hbar^2(x^{-2}+y^{-2})$, respectively. We show below that these correction terms are due to quantum effects that arise in the discussed transformations.

Let us, therefore, consider a Hamiltonian which contains an additive term,

$$H = H_0 - gF. \tag{1}$$

We assume that this (N-dimensional) Hamiltonian is integrable for every value of the coupling constant g (and of course for every value of the energy h), i.e., there exists a system of constants of motion in involution I_k, $k=1,\dots,N$. The noncanonical transformation consists of taking

$$G = H_0/F - h/F \tag{2}$$

as the new Hamiltonian. In effect we have just solved for the coupling constant and given it the role of the energy, hence the name "coupling constant metamorphosis." [Most of the time the new Hamiltonian G is very complicated, but next one can try to use canonical transformations to simplify it. In rare cases one obtains again a standard type Hamiltonian as will be shown below.]

The main result that will be used later is that the system (2) defined by the Hamiltonian G is also integrable for every value of the coupling constant h and, of course, for every value of the energy g. The constants of motion that guarantee integrability are, essentially, the I_k's. More precisely the I_k's defined for system (1) depend explicitly on g and to obtain the energy-independent constants of motion J_k for the new system (2) it is sufficient to substitute $H_0/F - h/F$ for g in the I_k's. It is easy to show that the J_k's so obtained are constants of motion and in involution. In fact one can prove the following very general proposition: Given any function $K = K(x_i, p_i)$, its time derivative under the Hamiltonian G is related to its time derivative under H by

$$(dK/dt)_G = F^{-1}(dK/dt)_H. \tag{3}$$

Therefore, an alternative way to look at this noncanonical transformation is as a change in the time variable. Introducing a time T by

$$dT = F\,dt, \tag{4}$$

we can say that if H describes the evolution in terms of t, G describes the same motion in terms of T.

As we have commented before, the Hamiltonian G can have a most unusual expression. However, in some cases suitable canonical transformations can reduce it to a standard form. In that case one establishes a duality between the Hamiltonians H and G. Although no general procedure exists for this, several interesting cases can be found.

a. Duality between the Hénon-Heiles and Holt Hamiltonians.—This was originally found by singularity study,[8] but now we can give explicitly the transformation connecting them. Let us start with the Holt Hamiltonian

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + \tfrac{3}{4}\mu x^{4/3} + (y^2 - g)x^{-2/3}. \tag{5}$$

This Hamiltonian is integrable for $\mu = 1$, 6, 16, and g free.[6,8] The noncanonical transformation leads to the Hamiltonian

$$G = \tfrac{1}{2}x^{2/3}(p_x^2 + p_y^2) + \tfrac{3}{4}\mu x^2 + y^2 - hx^{2/3}. \tag{6}$$

Making now the canonical reflection $p_y = Y$, $y = -P_Y$ and rescaling allows us to rewrite G as

$$G = \tfrac{1}{2}x^{2/3}p_x^2 + \tfrac{1}{2}P_Y^2 + \tfrac{3}{4}\mu x^2 + x^{2/3}y^2 - hx^{2/3}. \tag{7}$$

Using finally the canonical point transformation $x = (\frac{2}{3}X)^{3/2}$, $p_x = P_X(\frac{2}{3}X)^{-1/2}$, we obtain

$$G = \tfrac{1}{2}(P_X^2 + P_Y^2) + \tfrac{2}{3}(\tfrac{1}{3}\mu x^3 + xy^2) - \tfrac{3}{2}hx, \tag{8}$$

which is precisely the Hénon-Heiles Hamiltonian (with an additive term proportional to x) which, as is well known, is integrable for $\mu = 1$, 6, 16.[11,12]

b. The dual of the Fokas-Lagerstrom potential.—The Fokas-Lagerstrom Hamiltonian

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) - \tfrac{2}{3}g(xy)^{-2/3} \tag{9}$$

does not possess any free additive terms. In this case the duality transformation operates on the coupling constant g of the potential itself:

$$G = \tfrac{3}{4}(xy)^{2/3}(p_x^2 + p_y^2) - \tfrac{3}{2}h(xy)^{2/3}. \tag{10}$$

Next we make the same point transformation which led from (7) to (8), but now also for the y variable, to obtain

$$G=\tfrac{1}{2}(YP_X^2+XP_Y^2)-\tfrac{2}{3}hXY. \qquad (11)$$

This is now quadratic in X,Y suggesting the canonical transformation $X=(p_u+ip_v)/\sqrt{2}$, $Y=(p_u-ip_v)/\sqrt{2}$, $P_X=(-u+iv)/\sqrt{2}$, $P_Y=(-u-iv)/\sqrt{2}$, which leads after scaling to

$$G=\tfrac{1}{2}(p_u^2+p_v^2)-i(3h)^{1/2}/6[(u^2-v^2)p_u-2uvp_v]. \qquad (12)$$

Finally a canonical gauge transformation $p_u=P_u-i(3h)^{1/2}/6(u^2-v^2)$, $p_v=P_v-[i(3h)^{1/2}/3]uv$ gives the Hamiltonian

$$G=\tfrac{1}{2}(P_u^2+P_v^2)+\tfrac{1}{24}h(u^2+v^2)^2. \qquad (13)$$

This means that the Fokas-Lagerstrom potential is dual to the axially symmetric potential ρ^4!

To see how the second invariant changes one just makes the substitution (6) or (10) for g in I_2 and then one of the above sequences of canonical transformations. As a consequence the order (in p) of the second invariant changes and this explains the relationship between the orders of the invariants of the Hénon-Heiles and Holt Hamiltonians, which has been observed before.[8]

Let us now turn to the singularity structure of the above Hamiltonians. In Ref. 6 we have shown that the Holt and Fokas Hamiltonians are of the weak-Painlevé type,[5] i.e., their movable singularities are branch points of the form

$$x,y\sim(t-t_0)^{p/q}\sum_{n=0}^{\infty}\alpha_n(t-t_0)^{n/q}. \qquad (14)$$

However, as we have seen, these Hamiltonians are dual to the Hénon-Heiles and ρ^4 which are full Painlevé.[11] In our earlier work we have argued that weak-Painlevé Hamiltonians cannot be transformed into full-Painlevé ones by changes of the dependent or independent variables. However, the only independent-variable transformations we accepted had to be the same for every trajectory, no matter what the initial conditions were, and thus independent of the positions of the singularities. This does not hold for the "new-time" transformation (4), as T is now defined in terms of t and the x_i's and this is not, then, what one usually calls a change of the independent variable.

Let us see in detail how things work for the Fokas Hamiltonian (9). The noncanonical transformation allows us to cast it in the form (11) (let us take $h=\tfrac{3}{2}$ for simplicity). The corresponding equations of motion become thus

$$\ddot{X}=XY/Y+Y^2-\tfrac{1}{2}\dot{Y}Y^2X^{-2},$$
$$\ddot{Y}=XY/X+X^2-\tfrac{1}{2}\dot{X}X^2Y^{-2}. \qquad (15)$$

The only singular behavior of the solutions is in this case

$$X=A(T-T_0)^{-2}+\ldots,$$
$$Y=B(T-T_0)^{-2}+\ldots, \qquad (16)$$

with $(B/2)^3=1$ and $A=\tfrac{1}{2}B^2$. The resonances [defined as the orders of $(t-t_0)$ where arbitrary parameters can enter in the expansion] of Eq. (15) can be easily found; they are -1, 2 (double), and 3, and, moreover, the resonance conditions are satisfied. Thus (11) is of the full-Painlevé type.

In Ref. 6 we have investigated the singular expansions of the original Fokas Hamiltonian. Two such expansions were identified: one where x and y start as $(t-t_0)^{3/5}$ with all powers of $(t-t_0)^{1/5}$ entering in the expansion, and a second where x and y behave as

$$x\sim(t-t_0)^{3/4}\sum_{n=0}^{\infty}a_n(t-t_0)^{n/2},$$
$$y\sim A+B(t-t_0)+\sum_{n=0}^{\infty}b_n(t-t_0)^{n/2}. \qquad (17)$$

As we stated before, the noncanonical transformation can be interpreted as a transformation to a new time T defined by (4), or in this case

$$dT=(xy)^{-2/3}\,dt. \qquad (18)$$

For the first kind of singularity we have $dT\propto(t-t_0)^{-4/5}dt$, i.e., $(T-T_0)\propto(t-t_0)^{1/5}$, so that X and Y have expansions in powers of $T-T_0$. For the second kind of singularity we have $dT\propto(t-t_0)^{-1/2}\,dt$, i.e., $T-T_0\propto(t-t_0)^{1/2}$. Again X and Y have expansions in powers of $T-T_0$ (ignoring global factors). Moreover, both singular expansions for H correspond to regular expansions for G, and conversely the singular expansions (16) stem from a regular expansion for H.

It must be stressed, once again, that this conversion from weak to full Painlevé is achieved through a noncanonical transformation. The latter introduces a new time scale given by $dT=F(x,y)\,dt$ which varies from one trajectory to another and in the above cases is automatically tuned at each singularity so as to render it of Painlevé type. However, this does not work for all weak-Painlevé models, e.g., the polynomial potentials of Ref. 5 remain weak Painlevé after coupling constant metamorphosis with respect to the most obvious additional terms.

Let us discuss the quantum effects of the above transformations. It is well known that even ordinary point canonical transformations are very complicated in quantum mechanics. For quantum integrability we have used an algebraic approach with c-number functions and Moyal brackets.[10] This is closely related to the path-integral approach to quantum mechanics, where the problems of point canonical transformations were solved some time ago.[13] It was found, e.g., that in two dimensions the change from Cartesian to polar coordinates generates a correction term $-\frac{1}{8}\hbar^2 r^{-2}$, (which appears in the angular momentum quantization rule $L^2=m^2-\frac{1}{4}$).

In the present problem we also have correction terms but now the transformation is more complicated. Fortunately also the "new-time" transformation has recently been introduced for path integrals,[14,15] where it is used together with a point canonical transformation, as was done above.

Suppose, then, that we want to make a canonical point transformation $q=f(Q)$, $p=P/f'(Q)$ to the Hamiltonian $\frac{1}{2}p^2+V(q)$ and then bring the kinetic energy to the standard form by multiplying the Hamiltonian by $f'(Q)^2$. The net quantum correction[15] that should be added to the potential can be written in the form

$$\Delta_{\text{total}} V(Q)=-\frac{1}{4}\hbar^2\left\{\frac{d}{dQ}\left(\frac{f''}{f'}\right)-\frac{1}{2}\left(\frac{f''}{f'}\right)^2\right\}. \quad (19)$$

It is interesting to note that this is nothing but the "Schwarzian derivative" that has appeared recently in some other integrability studies.[16]

Let us now apply this to the present case. Going backwards from Eq. (8) to Eq. (5) the only quantum effects arise precisely from a canonical point transformation and the ensuing overall multiplication. In this case $f(X)\propto X^{2/3}$, and it is easy to see from (19) that then

$$\Delta_{\text{total}} V(X)=-\tfrac{5}{72}\hbar^2 X^{-2}. \quad (20)$$

This is precisely the additional term that was found before.[5,8] Similarly to get from (13) to (9) the steps from (13) to (11) do not generate any corrections, but from (11) to (9) a transformation $f(Q)\propto Q^{2/3}$ and multiplication is needed for both the x and y variables. This yields the correction term $-\frac{5}{72}\hbar^2(x^{-2}+y^{-2})$, which is the deformation that was found necessary for quantum integrability of the Fokas-Lagerstrom potential.[7,8]

In the above examples we were able to find the sequence of transformations which eliminated the correction terms that were needed for quantum integrability. From a different point of view the existence of such a correction term can be taken as an indication that there is another formulation (different coordinates, coupling constants) where no such terms are needed, and which therefore is a more natural one. This change of formulation can be useful even in classical mechanics although the signal for it appears explicitly only in quantum mechanics. The aim of the present work is to show through examples how one can exploit the existence of a free coupling constant in integrable Hamiltonians to obtain new integrable systems. The cornerstone of our method was a noncanonical transformation which converts the coupling constant into energy and vice versa. When it is combined with canonical transformations it can connect Hamiltonians of various standard forms. In the cases of Holt and Fokas-Lagerstrom potentials it allows the conversion from weak to full Painlevé, and it explains the correction terms that were found necessary for their quantum integrability. We believe that these rich implications of the presence of additive terms in integrable systems, combined with the coupling-constant metamorphosis, will stimulate further search for such terms in the cases where they are not known yet and spur the study of integrable systems in general.

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