Universal Power Law for the Dimension of Strange Attractors near the Onset of Chaos

In a recent Letter[1] it was shown that the envelope of the Lyapunov exponent of one-dimensional, one-parameter familes of dynamical systems that period double on their way to chaos rises like $(\mu-\mu_\infty)^\beta$ where μ is the stress parameter, μ_∞ is the point of accumulation of period doublings, and $\beta=\ln 2/\ln\delta$, $\delta=4.669.\ .\ .\ .$ Here we comment that (i) this result can be generalized to the (single) positive Lyapunov exponent in any dissipative multidimensional map $M_\mu:R^F\to R^F$ that undergoes period doubling, and (ii) that this implies a universal scaling law for the dimension of strange attractors which develop after the completion of a cascade of period doublings. By this we mean that the information dimension[2] D_1 depends on the stress parameter μ according to $(D_1-1)\sim(\mu-\mu_\infty)^\beta$ for values of μ that do not belong to a window of periodicity.

The vector of Lyapunov exponents, $\vec{\lambda}[M_\mu]$ obeys the identity $\vec{\lambda}[M_\mu]=2^{-n}\vec{\lambda}[T^nM_\mu]$, where T is the doubling operator.[3] For $\mu-\mu_\infty<<1$ one can linearize T and show that $\vec{\lambda}[M_\mu]=2^{-n}\vec{\lambda}[G+\delta^n(\mu-\mu_\infty)aH]=(\mu-\mu_\infty)^\beta\vec{\lambda}[G+aH]$. Here G is the fixed-point map of T at μ_∞, a is a number, and H is the eigenfunction in the (single) unstable direction. The map G has $F-1$ (negative) infinite Lyapunov exponents due to infinite contraction rates in $F-1$ directions.[3] Accordingly, $F-1$ Lyapunov exponents of M_μ may reach a (negative) finite limit (a "regular part") when $\mu=\mu_\infty$ and only the one associated with the singled out direction in phase space[3] vanishes like $(\mu-\mu_\infty)^\beta$ (i.e., is purely "singular").

To calculate the dimension, we invoke Kaplan-Yorke (KY) formula[4,5] $D_1=j+\ \sum_{i=1}^{j}\lambda_i/|\lambda_{j+1}|$, where j is the highest index for which $\sum_{i=1}^{j}\lambda_i\geq 0$. Since $|\lambda_{j+1}|$ would be dominated by its regular part, the considerations above lead directly to the *universal* scaling law of (D_1-1).

We tested the prediction numerically in two examples, i.e., Helleman's standard map $(x',y')=(2\mu x+2x^2+by,x)$, $b=0.3$, $\mu_\infty=-0.7639$, and the simple three-dimensional map $(x,y,z)=(1-\mu x^2+by+cz,x,y)$, $b=c=0.15$, $\mu_\infty=1.3892$. Excellent agreement with the scaling law of (D_1-1) is found for all values of μ that do not belong to a window of periodicity; see Fig. 1.

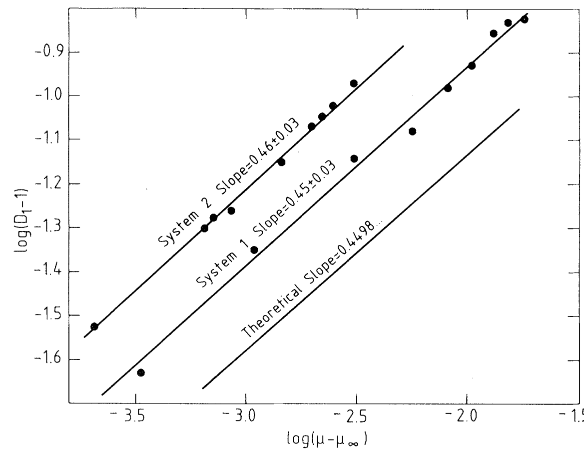

FIG. 1. $\ln(D_1-1)$ vs $\ln(\mu-\mu_\infty)$ for the examples discussed in the text.

An experimental verification of the scaling law of (D_1-1) is particularly important since it would serve as a test of the KY formula in addition to providing support for the theory leading to this scaling law. Such a test is important since a rigorous proof of the KY formula as an equality for high-dimensional systems is still lacking, although insight on its validity was obtained in Ref. 5.

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[1]B. A. Huberman and J. Rudnick, Phys. Rev. Lett. **45**, 154 (1980).

[2]H. G. E. Hentschel and I. Procaccia, Physica (Utrecht) **D8**, 435 (1983).

[3]P. Collet, J. P. Eckman, and H. Koch, J. Stat. Phys. **25**, 1 (1981).

[4]J. L. Kaplan and J. A. Yorke, in *Functional Differential Equations and Approximation of Fixed Points,* Springer Lecture Notes in Mathematics Vol. 730, edited by A. Dold and B. Eckman (Springer-Verlag, Berlin, 1978), p. 228.

[5]P. Grassberger and I. Procaccia, to be published.

