Chiral Anomaly and Quantized Hall Effect

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Chiral anomaly in (2+1)-dimensional fermion theory coupled with an external electromagnetic field is studied. Its connection with the quantized Hall effect is pointed out. Hall conductivity is quantized and agrees with the experimental value.

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In a recent paper,[1] it is pointed out that chiral anomaly is due to the appearance of a path-dependent phase factor in the Euclidean partition function and a path-dependent term in the logarithm of the partition function, the free energy. The path-dependent phase factor is produced by a zero-eigenvalue solution of the Euclidean Dirac equation. Since the zero eigenvalue of the Dirac equation makes the partition function vanish, and the free energy become infinity, this is a singular point of the partition function. The path-dependent phase factor can be produced by the singularity but not by the regular part, which is the ordinary part in the Lagrangian. Hence there is no correction from higher-order effects in the anomaly term.

Here we shall discuss the same anomaly in 2+1 dimensions.[2] Actually the anomaly in 2+1 dimensions has been obtained already,[3] but it may be useful to discuss it from a different point of view.

A consistency condition for an external magnetic field which couples with fermions and a current which is induced by an external electric field shall be obtained. The induced current which is orthogonal to the electric field is shown to exist. Its value remains constant in the large fermion mass limit, and in the small mobility limit, and agrees with the experimentally observed quantized Hall current.

We investigate a system described by

$$\mathscr{L}=\bar{\psi}\gamma^{\mu}(i\hbar\partial_{\mu}+eA_{\mu})\psi-m\bar{\psi}\psi, \tag{1}$$

in 2+1 dimensions. In these dimensions, γ_μ matrices are 2×2 and satisfy

$$\epsilon_{\mu\nu\rho}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}=i\ (1), \tag{2}$$

in Minkowski (Euclidean) space. Here $\epsilon_{\mu\nu\rho}$ is a totally antisymmetric tensor.

A peculiar property which has its origin in Eq. (2) and will be used later can be seen in the quantity $\sum_{\text{all } n}\mathrm{Tr}[\phi_n^\dagger(\vec{x})\gamma_0\phi_n(\vec{x})]$, where $\{\phi_n(\vec{x})\}$ is a complete set satisfying

$$\bar{\not{D}}\phi_n=i\sum_{j=1}^{2}\gamma^j(i\hbar\partial_j+eA_j)\phi_n=\lambda_n\phi_n. \tag{3}$$

Not only is the calculation of $\sum_n\mathrm{Tr}[\phi_n^\dagger(\vec{x})\gamma_0\times\phi_n(\vec{x})]$ itself important, but also similar calculations shall be needed several times in this paper, and so I show it here. The method is due to Fujikawa[4]:

$$\begin{aligned}\sum_{\text{all } n}\mathrm{Tr}[\phi_n^\dagger(\vec{x})\gamma_0\phi_n(\vec{x})]&=\lim_{M\to\infty}\sum_n\mathrm{Tr}[\phi_n^\dagger(\vec{x})\gamma_0\exp(-\lambda_n^2/M^2)\phi_n(\vec{x})]\\&=\lim_{M\to\infty}\sum_n\mathrm{Tr}[\phi_n^\dagger(\vec{x})\gamma_0\exp(-\bar{\not{D}}^2/M^2)\phi_n(\vec{x})]\\&=\lim_{M\to\infty}\sum_k e^{-ikx}\mathrm{Tr}\gamma_0\exp(-\bar{\not{D}}^2/M^2)e^{ikx}\\&=\lim_{M\to\infty}\sum_k e^{-ikx}\mathrm{Tr}\gamma_0\exp\{-M^{-2}(D_\mu D^\mu+\tfrac{1}{2}[\gamma^\mu,\gamma^\nu]F_{\mu\nu})\}e^{ikx}\\&=(2\pi\hbar)^{-1}\exp(F_{12}).\end{aligned} \tag{4}$$

In the above equation, the regularization mass M was introduced in the intermediate step. The final result is, however, independent of M and this term exists even in the finite-M case.[5] The zero-eigenvalue solution of Eq. (3) $(\lambda=0)$[6] contributes to this term. In fact, $\int d^2x\,\mathrm{Tr}[\phi_n^\dagger(\vec{x})\gamma_0\phi_n(\vec{x})]$ is nonzero only for the zero-eigenvalue solution, $\phi_0(\vec{x})$, since γ_0 anticommutes with $\bar{\not{D}}$.

We now obtain one consistency condition for a magnetic flux, based on a representation of the partition function Z, and an induced current, based on a different representation.

The Green's function in Euclidean space is calculated from

$$Z=\int d\mu_F\exp[-\hbar^{-1}\int\mathscr{L}\,d^3x]=\int d\mu_F\exp[-\hbar^{-1}\int dx_0\{b_n h_{nm}a_m'+(i\lambda_m'+m)b_na_n\}], \tag{5}$$

where

$$d\mu_F = \prod da_n(x_0)\, db_m(x_0), \quad \psi(x_0,\vec{x}) = \sum\nolimits_n a_n(x_0)\phi_n'(\vec{x}), \quad \bar{\psi}(x_0,\vec{x}) = \sum\nolimits_n b_n(x_0)\phi_n'(\vec{x}),$$
$$h_{nm} = \int d^2x\, \phi_n'^{\dagger}(\vec{x})\gamma_0\phi_m'(\vec{x}), \quad [\gamma^0 eA_0 + \sum_{j=1}^{2}\gamma^j(i\hbar\partial_j + eA_j)]\phi_n'(\vec{x}) = i\lambda_n'\phi_n'(\vec{x}). \tag{6}$$

The external vector fields A_μ are assumed to be time independent in this part.

Let us study the following transformation:

$$\int dx_0\psi(\vec{x},x_0) \rightarrow e^{i\theta(\vec{x})\gamma_0}\int dx_0\psi(\vec{x},x_0), \quad \int dx_0\bar{\psi}(\vec{x},x_0) \rightarrow \int dx_0\bar{\psi}(\vec{x},x_0)e^{i\theta(\vec{x})\gamma_0}. \tag{7}$$

The measure $d\mu_F$ is transformed as

$$\prod da_n(x_0)\, db_m(x_0) \rightarrow \prod da_n' db_m' = \det C^2 \prod da_n\, db_m, \tag{8}$$

where $C_{nm} = (\phi_n'^{\dagger}e^{i\theta\gamma_0}\phi_m)$. Following Fujikawa,[4] we calculate $\det C^2$ for infinitesimal θ as

$$\det C^2 = \exp \operatorname{tr}\ln C^2 = \exp[\sum\nolimits_n 2i(\phi_n'^{\dagger}i\theta\gamma^0\phi_n')] = \exp[2i(e/h)\int d^2x\, F_{12}\theta(\vec{x})]. \tag{9}$$

Hence, for finite $\theta(\vec{x})$ we have

$$\prod \det C^2 = \exp[2i(e/h)\int d^2x\, F_{12}\theta(\vec{x})]. \tag{10}$$

From the transformation Eq. (7), the field ψ is transformed back to the original field if $\theta(\vec{x})$ is equal to 2π. Thus the physical requirement that Z is single valued[7] leads to

$$\exp[2i\int d^2x\, 2\pi(e/h)F_{12}] = 1, \quad \int d^2x\, F_{12} = \tfrac{1}{2}(h/e)\times\text{integer}. \tag{11}$$

This is a flux quantization condition.

Next we study the change of action which is caused by an adiabatic change of A_μ. It is convenient to calculate in Euclidean space. Since Z is equal to the determinant of $\not{D} - m$, its small change is given by

$$\delta Z = Z\operatorname{Tr}[(\not{D} - m)^{-1}\gamma_\mu\delta A^\mu], \tag{12}$$

when A_μ is changed adiabatically by δA_μ. Thus

$$\begin{aligned}\delta\ln Z &= \operatorname{Tr}[(\not{D} - m)^{-1}\gamma_\mu\delta A^\mu] = \operatorname{Tr}\{(\not{D} + m)[(\not{D} - m)(\not{D} + m)]^{-1}\gamma_\mu\delta A^\mu\}\\ &= \int_{-\infty}^{0} dS\operatorname{Tr}\{(\not{D} + m)e^{S(\not{D}^2 - m^2)}\gamma_\mu\delta A^\mu\} = (e^2/4\pi\hbar)\epsilon^{\mu\nu\rho}F_{\mu\nu}\delta A_\rho + m^{-2}O(F^2)\delta A.\end{aligned} \tag{13}$$

In the last step of the above equation, use has been made of a method similar to Eq. (4). The first term corresponds to the topological term. The coefficient is uniquely determined.

From Eq. (13), we have an induced current:

$$\frac{\delta\ln Z}{\delta A_i} = \frac{e^2}{4\pi\hbar}\epsilon^{i\nu\rho}F_{\nu\rho} + \frac{1}{m^2}O(F^2) = \frac{e^2}{h}F_{0j} + \frac{1}{m^2}O(F^2) \quad (j \neq i). \tag{14}$$

Now we apply the previous argument to planar electrons under strong magnetic field. The Hall effect will be studied.

The electron's energy is small in solid state physics and the relativistic effect is negligible. The Schrödinger equation may be used without any serious problems. However, in some situations two states are involved in the system's time development and a momentum-dependent correlation between them exists. I would like to propose the use of the Dirac equation in this case.

Let us assume that there are only two states, which are represented by fields $\psi_1(x)$ and $\psi_2(x)$ and have masses m_1 and m_2. Furthermore, mobility is assumed to exist. Although the physical reason for the mobility in two-dimensional systems has not been made clear, we can study its physical consequences and obtain a universal property about Hall conductivity based on this assumption. Even in the limit of vanishing mobility the result regarding Hall conductivity is unchanged.

The Hamiltonian density which describes two electron states is assumed to be[8]

$$\mathscr{H} = m_1\psi_1^{\dagger}(x)\psi_1(x) + m_2\psi_2^{\dagger}(x)\psi_2(x) + \{c_x\psi_1^{\dagger}i\hbar\partial_x\psi_2 + c_y\psi_1^{\dagger}i\hbar\partial_y\psi_2 + \text{H.c.}\}. \tag{15}$$

It is assumed that their momenta are small and only linear terms with respect to momentum are included. The coefficients c_i are complex constants, which may be very small.

It is convenient to start from the Lagrangian when we investigate the electromagnetic properties of the system. The Lagrangian density by which the previous Hamiltonian is derived is

$$\mathscr{L} = \bar{\psi}[\gamma_0(i\hbar\partial_0 + e_0) - d_i^j\gamma^i(i\hbar\partial_j) - m]\psi, \tag{16}$$

where

$$\bar{\psi} = (\psi_1^\dagger, \psi_2^\dagger)\gamma_0, \quad \psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad m = m_1 - m_2, \quad e_0 = \tfrac{1}{2}(m_1 + m_2),$$
$$(d) = \begin{pmatrix}\mathrm{Im}c_x & -\mathrm{Re}c_x\\ \mathrm{Im}c_y & -\mathrm{Re}c_y\end{pmatrix}, \quad \gamma_0 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \gamma_1 = i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \gamma_2 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \tag{17}$$

after a suitable phase convention is chosen. The coupling of ψ with an external electromagnetic field is determined by the usual minimal coupling. Thus we have

$$L = \bar{\psi}[\gamma_0(i\hbar\partial_0 + eA_0 + e_0) + d_i^j\gamma^i(i\hbar\partial_j + eA_j) - m]\psi. \tag{18}$$

In a situation where the two states are generated by an external magnetic field special attention is needed. A strong magnetic field in an orthogonal direction to the planar electrons is known to lead to the electrons splitting into many levels which are confined spatially and are called Landau levels. A vector potential $(-\frac{1}{2}(y-Y)H, \frac{1}{2}(x-X)H) = (A_x, A_y)$, which corresponds to a constant magnetic field, leads to the Landau-level center (X, Y) for instance. When we study the response of the system to an additional external change, we add the additional term to the vector potential.[9] Only these terms should be put into Eq. (18) in order to avoid double counting. More specifically, if a time-independent small current is added to the system, the response to it is studied by adding $(\delta A_0, \delta A_1, \delta A_2)$ to the vector potential. Here δA_1 and δA_2 are connected with the current density through

$$\delta W/\delta A_i = j_i, \tag{19}$$

where δW is the corresponding change of action. We understand that the vector potential in Eq. (18) means these potentials hereafter.

It would be reasonable to assume that only the nearest-neighbor two levels among many levels are involved in low-energy phenomena if all energy levels are well separated and if the Fermi energy is located in their gap region. This may occur under strong magnetic field. Thus the phenomenon which we are discussing may occur only under strong magnetic field.

The effect of finite electron density is represented by inclusion of the chemical potential in the Lagrangian. We do not need to change anything except to redefine e_0 in Eq. (17).

The induced current density is seen from Eq. (14). The first term is unchanged with the presence of the numerical constant d_j in Eq. (18).[5] Even in the limit of vanishing mobility ($c_j \to 0$), the value is unchanged. The second term, however, vanishes if c_j vanishes. The localization of charge carriers corresponds to the vanishing of c_j. Even in that limit, there is a current, with quantized value, in the direction orthogonal to the electric field. The current in the direction parallel to the electric field vanishes in that limit.

If there are many levels below the Fermi energy, a unit of quantized current from each gap region contributes. Thus the total current density is given by

$$j_i = (e^2/h)F_{0j} \times \text{integer} \quad (j \neq i). \tag{20}$$

Recently, quantization of the Hall current was measured by Klitzing, Dorda, and Pepper.[10] The observed Hall conductivity is

$$(e^2/h) \times \text{integer}, \tag{21}$$

when the conductivity in the direction parallel to the current vanishes. The value calculated here agrees with the experimentally observed value.

In the present argument the ψ may be any bound states, since we do not need any property of ψ except that they can move and have unit charge. They do not need to be pure Landau levels, but may be those states which are modified by phonon interactions.

Laughlin[11] gave a general argument based on gauge invariance. Ours may be a microscopic extension of Laughlin's argument, although there seems to be a distinction between them.

Note added.—This is a revised version of Hokkaido University Report No. EPHOU 83 DEC 005.

The derivation of the induced current is slightly changed. After completion of the present Letter I became aware of papers which discuss similar topics: Y. N. Srivastava and A. Widom, to be published; M. H. Friedman *et al.*, Phys. Rev. Lett. **52**, 1587 (1984).

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[5]It is important to note that a modified operator $i\sum_{j=1}^{2} d_j^i \gamma^j (i\hbar\partial_i + eA_i)$ with nonsingular d_j^i leads to the same result as Eq. (4).

[6]Although this eigenvalue is the one of $\bar{\not{D}}$ in Eq. (3) and different from the one of the Euclidean Dirac operator $i\sum_\mu \gamma^\mu (i\hbar\partial_\mu + eA_\mu)$, there is a deep connection between them. Since the normalized zero-eigenvalue solution of the Euclidean Dirac operator satisfies

$$\left[\frac{\partial}{\partial x_0} - \left\{\gamma^0\gamma^i\left(i\frac{\partial}{\partial x_i} + eA_i\right) + eA_0\right\}\right]\phi = 0,$$

an adiabatic treatment of the energy operator $\gamma^0\gamma^i \times (i\partial/\partial x_i + eA_i) + eA_0$ leads the eigenvalue to change sign and cross zero. Conversely, a suitable value of A_0 for a given A_i leads an energy eigenvalue to change sign. Thus a normalizable zero-eigenvalue solution to the Euclidean Dirac operator exists in this case.

[7]If Z were not single valued, it would vanish because of a destructive interference. This may suggest that no ground state can be defined in this case. The situation is the same as that of the electron system around a magnetic monopole which does not satisfy the well-known quantization condition.

[8]Both center coordinates of the Landau levels are chosen to be diagonal in our representation. The formalism is similar to but different from that of T. Ando and Y. Uemura, J. Phys. Soc. Jpn. **36**, 959 (1974); R. Kubo, S. J. Miyake, and N. Hashitsume, in *Solid State Physics: Advances in Research and Applications,* edited by F. Seitz and D. Turnbull (Academic, New York, 1965), Vol. 17, p. 259. Only the term linear with respect to the momentum, which is generated by scatterers, is taken. A more detailed account of the representation will be presented elsewhere.

[9]The change of the external field may lead the internal property of the field ψ to be changed. However, as long as the change is made with a large scale (long wavelength), the internal change of ψ may be irrelevant to the large-scale property of the system.

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