Relativistic Naive Quark Model for Spinning Quarks in Mesons

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We use two-body Dirac equations (derived from Dirac's constraint mechanics and supersymmetry) to make the naive quark model fully relativistic. Our covariant equations incorporate not only relativistic kinematics but also dynamical recoil effects that generalize Breit corrections to forms that have well-defined quantum short-distance behavior. In these equations, a crude but covariant generalization of Richardson's static quark potential produces a surprisingly good one-parameter relativistic fit to the meson spectrum.

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The nonrelativistic naive quark model provides a good first approximation to the quantum dynamics of heavy quarks in mesons. We propose to extend its simple dynamical structure to the light-quark mesons through a relativistic description of two interacting spinning particles given by two coupled, compatible Dirac equations on a sixteen-component wave function.[1] This exactly relativistic treatment of the two-body problem, which follows from a supersymmetric extension of Dirac's constraint mechanics,[2] avoids classical no-interaction theorems and quantum ghosts through covariant control of the relative time. The two-body Dirac equations reduce to a Schrödinger-like four-component form that combines strong-potential structure important in the Dirac phenomenology of nuclear physics[3] with recoil structure appropriate to a relativistic two-body system. This new structure, necessary for interacting Dirac particles, is missing for one or both particles from previous works that relied on semirelativistic (slow-motion, weak-potential) approximations (using Breit-corrected Schrödinger equations[4]), partly relativistic schemes that treated the kinematics exactly while approximating the potential as though it were weak (using truncations of the Bethe-Salpeter equation or the full Breit equation), or asymmetric relativistic schemes that hold one particle on mass shell.[5] The neglect of this structure has led some authors to overestimate the size of many-body and chiral-symmetry contributions for the light mesons.[6] To test the importance of this new nonperturbative structure in our equations for relativistic quark-antiquark interactions we calculate the meson spectrum predicted by inserting into our equations a one-parameter relativistic generalization of Richardson's static heavy-quark potential.[7] Previously[8] we extended this potential to the light- and intermediate-mass vector mesons in a two-body Klein-Gordon equation derived by means of Dirac's constraint mechanics.[2]

Richardson's non-relativistic potential takes the form $V(r)=8\pi\Lambda^2 r/27-8\pi f(\Lambda r)/27r$, combining asymptotic freedom and linear quark confinement [since for $r\rightarrow 0$, $f(\Lambda r)\rightarrow -1/\ln\Lambda r$, while for $r\rightarrow\infty$, $f(\Lambda r)\rightarrow 1$]. Its relativistic generalization in the constraint approach requires two important modifications. First, compatibility of the constraints implies that the variable r must be reinterpreted covariantly as the interparticle separation in the c.m. system. That is, $r=(x_\perp^2)^{1/2}$ where $x_\perp^\mu=(g^{\mu\nu}-P^\mu P^\nu/P^2)(x_1-x_2)_\nu$, with P the total momentum. Secondly, Richardson's potential must be reinterpreted as the static part of a system of covariant potentials. Although there is no field-theoretical justification for treating the long-range part as a scalar, string arguments[9] indicate that there are no long-range $(1/r)$ spin-spin terms so that the long-range potential cannot be an electromagnetic vector with both spacelike (magnetic) and timelike (electrostatic) parts. Hence some authors assume that the long-range part should be treated purely as a timelike four-vector[4] [here of the form $V^\mu=\mathscr{V}\hat{P}^\mu$, where $\hat{P}^\mu=P^\mu/w$ and $w=(-P^2)^{1/2}$ is the c.m. total energy]. Just as does the squared form of the Dirac equation, our two-body form contains a nonconfining $-\mathscr{V}^2$ piece produced by the long-range electrostatic part. Since the scalar interaction leads to a corresponding contribution of $+S^2$ in such a model, we must include a long-range scalar part to maintain confinement. To avoid adding an extra parameter to our potential model, we divide the long-range part into half scalar and half timelike four-vector while assuming the shorter-range Coulomb-like piece to be an electromagneticlike four-vector given in terms of a scalar $\mathscr{A}$. That is, $S=\mathscr{V}=8\pi\Lambda^2 r/54$ while $\mathscr{A}=-8\pi f(\Lambda r)/27r$. This choice leads to a cancellation of spin-orbit effects at long range, thus preventing partial multiplet inversions for the lighter mesons.

In the spinless case, these potential forms are determined by requiring the desired nonrelativistic limit and compatibility of the covariant generalized mass shell constraints for the constituent particles, $\mathscr{H}_i=(p_i-A_i)^2+(m_i+S_i)^2\approx 0$, in which constituent vector potentials given by $A_i^\mu=\alpha_i p_1^\mu+\beta_i p_2^\mu$ are introduced by minimal substitutions. The classical compatibility condition on the two constraints, $\{\mathscr{H}_1,\mathscr{H}_2\}\approx 0$, implies that only two of the four scalars α_i,β_i and one of the two scalars S_i are independent. These three independent system scalars, $\mathscr{A}$, $\mathscr{V}$, and S, must be functions of $x_\perp^\mu$ and are related to constituent potentials A_i^μ, S_i by $M_i^2\equiv[m_i+S_i(S,\mathscr{A})]^2=m_i^2+G^2(2m_w S+S^2)$, $\pi_i^\mu\equiv p_i^\mu-A_i^\mu=\hat{P}^\mu E_i(\mathscr{A},\mathscr{V})\pm Gp^\mu$, where $G^2\equiv 1/(1-2\mathscr{A}/w)$ and $E_i(\mathscr{A},\mathscr{V})\equiv G[(\epsilon_i-\mathscr{A})^2-2\epsilon_w\times\mathscr{V}+\mathscr{V}^2]^{1/2}$. The variables ϵ_1 and ϵ_2 are the c.m. energies of the constituent particles while the relative momentum is $p^\mu=(\epsilon_2 p_1^\mu-\epsilon_1 p_2^\mu)/w$; $m_w=m_1\times m_2/w$ and $\epsilon_w=(w^2-m_1^2-m_2^2)/2w$ are the relativistic reduced mass and energy. In the quantum case, with Hermitian ordered A_i^μ's, the two-body Klein-Gordon equations are $\mathscr{H}_i\psi=0$.

Even with potential forms fixed by requiring spinless compatibility, the transformation from the two-body Klein-Gordon equations to the two-body Dirac equations is not as simple as replacing each of the Klein-Gordon forms by the corresponding one-body Dirac form. In the spin-dependent case, compatibility of constituent Dirac equations requires the presence of extra spin-dependent recoil terms that have no one-body analog. These terms are consequences of supersymmetries of the pseudoclassical two-body Dirac system. The two-body (c.m.) Dirac equations with scalar[1] and both types of vector interaction are

$$\begin{aligned}\mathscr{S}_1\psi&=\gamma_{51}G\left(\vec{\gamma}_1\cdot\vec{\mathrm{p}}+\frac{(M_1-E_1\gamma_1^0)}{G}+\frac{i}{2}\vec{\gamma}_2\cdot\nabla\ln G\,\vec{\gamma}_2\cdot\vec{\gamma}_1+\frac{i}{2}\vec{\gamma}_2\cdot\frac{\nabla E_1}{E_2}\gamma_2^0\gamma_1^0-\frac{i}{2}\vec{\gamma}_2\cdot\frac{\nabla M_1}{M_2}\right)\psi=0,\\ \mathscr{S}_2\psi&=\gamma_{52}G\left(-\vec{\gamma}_2\cdot\vec{\mathrm{p}}+\frac{(M_2-E_2\gamma_2^0)}{G}-\frac{i}{2}\vec{\gamma}_1\cdot\nabla\ln G\,\vec{\gamma}_1\cdot\vec{\gamma}_2-\frac{i}{2}\vec{\gamma}_i\cdot\frac{\nabla E_2}{E_1}\gamma_1^0\gamma_2^0+\frac{i}{2}\vec{\gamma}_1\cdot\frac{\nabla M_2}{M_1}\right)\psi=0.\end{aligned}\tag{1}$$

The three extra spin-dependent terms at the ends of these equations vanish in either heavy-particle limit. With these extra recoil terms, the compatibility condition, $[\mathscr{S}_1,\mathscr{S}_2]=0$, is automatically satisfied for potentials obeying the restrictions already encountered in the bosonic case. We perform an exact reduction of the two sixteen-component wave equations (1) to the following four decoupled (with diagonal γ_i^0's) four-component Schrödinger-like equations (see Ref. 1 for details of this procedure):

$$(\vec{\mathrm{p}}^2+\Phi_{\mathrm{S.I.}}+\Phi_{\mathrm{S.S.}}+\Phi_{\mathrm{S.O.}}+\Phi_T+\Phi_{\mathrm{D.O.}})\psi=b^2\psi,\tag{2}$$

where

$$\Phi_{\mathrm{S.I.}}=2m_w S+S^2+2\epsilon_w\mathscr{V}-\mathscr{V}^2+2\epsilon_w\mathscr{A}-\mathscr{A}^2,$$

$$\Phi_{\mathrm{S.S.}}=-\nabla^2\ln(\chi_1\chi_2 G^{1-2\vec{\sigma}_1\cdot\vec{\sigma}_2/3})/2+[\nabla\ln(\chi_1\chi_2 G^{1-2\vec{\sigma}_1\cdot\vec{\sigma}_2/3})]^2/4+(\nabla\ln G)^2(3+\vec{\sigma}_1\cdot\vec{\sigma}_2)/18,$$

$$\Phi_{\mathrm{S.O.}}=-(\partial\ln\chi_1/\partial r\,\vec{\mathrm{L}}\cdot\vec{\sigma}_1+\partial\ln\chi_2/\partial r\,\vec{\mathrm{L}}\cdot\vec{\sigma}_2)/r,$$

$$\Phi_T=S_T[-(r\,\partial^2\ln G/\partial r^2-\partial\ln G/\partial r)/r+\nabla\ln G\cdot\nabla\ln(\chi_1\chi_2)]/6,$$

$$\begin{aligned}\Phi_{\mathrm{D.O.}}=(\vec{\mathscr{M}}-\vec{\mathscr{E}})^2/4-\{&\epsilon_2\,\vec{\sigma}_1\cdot\nabla[\vec{\sigma}_2\cdot(\vec{\mathscr{M}}-\vec{\mathscr{E}})]/w+\epsilon_1\,\vec{\sigma}_2\cdot\nabla[\vec{\sigma}_1\cdot(\vec{\mathscr{M}}-\vec{\mathscr{E}})]/w\\ &-\vec{\sigma}_1\cdot\vec{\sigma}_2\nabla\ln G\cdot(\vec{\mathscr{M}}-\vec{\mathscr{E}})-\vec{\sigma}_1\cdot\nabla\ln(\chi_1)\,\vec{\sigma}_2\cdot(\vec{\mathscr{M}}-\vec{\mathscr{E}})\\ &-\vec{\sigma}_2\cdot\nabla\ln(\chi_2)\,\vec{\sigma}_1\cdot(\vec{\mathscr{M}}-\vec{\mathscr{E}})\}(-)^s(\Phi_{\mathrm{S.I.}}-b^2)/(2\chi_1\chi_2),\end{aligned}$$

with $\chi_i=(E_i\gamma_i^0+M_i)/G$, $\vec{\mathscr{M}}=\nabla(M_1^2+M_2^2)/4M_1M_2$, and $\vec{\mathscr{E}}=\nabla(E_1^2+E_2^2)/4E_1E_2\gamma_1^0\gamma_2^0$.

The square of the c.m. on-shell value of the relative momentum has the effective one-body form $b^2=\epsilon_w^2-m_w^2$ building correct two-body relativistic kinematics into (2). The interaction Φ is divided into five parts. The spin-independent part $\Phi_{\mathrm{S.I.}}$, the spin-spin part $\Phi_{\mathrm{S.S.}}$ (including the Darwin terms), the spin-orbit part $\Phi_{\rm S.O.}$, and the tensor term Φ_T are accompanied by $\Phi_{\rm D.O.}$, the diagonalized form of the doubly odd (in $\vec{\gamma}_1$ and $\vec{\gamma}_2$) pieces stemming from the end terms in our two-body Dirac equations.

Although the potentials in (1) and (2) have a static appearance, the covariant equations in which they appear produce corrections agreeing perturbatively with dynamical recoil effects implied by relativistic quantum field theory.[10] We have found that if we let $S=\mathscr{V}=0$ and $\mathscr{A}=-\alpha/r$, (2), in its weak potential form $[\ln(1-A/2w)^{-1/2}\sim A/w]$ is equivalent to the Todorov equation for spin-one-half particles under mutual electromagnetic interaction.[11] Furthermore, the slow motion (semirelativistic) limit of this weak potential form is canonically equivalent[12] to the Fermi-Breit approximation to the Bethe-Salpeter equation, thus producing the usual spectral results through order α^4 for two-body bound states. The success of the constraint approach in predicting the α^4 (hyper)fine structure of QED with just the Coulomb potential as input gives us confidence that it may accurately reflect the (hyper)fine structure of QCD with an appropriate static potential as input. Unlike the local forms of the Bethe-Salpeter equation or even the Todorov equation, our Pauli forms make quantum-mechanical sense in the strong potential, nonperturbative regime where relativistic effects of the wave operator on ψ are not negligible. This claim is easiest to explain through the following comparison of the main spin-spin term of our equation with those of Breit and Todorov (two-body Dirac form $\rightarrow$ Breit form):

$$-\frac{1}{6}\sigma_1\cdot\sigma_2\nabla^2\ln\left|1-\frac{2\mathscr{A}}{w}\right|\rightarrow\frac{1}{3}\,\frac{\sigma_1\cdot\sigma_2\nabla^2\mathscr{A}}{m_1+m_2}.$$

For $\mathscr{A}$'s that have singular short-range behavior like $-\alpha/r$ (QED) and $8\pi/27r\ln r$ (QCD) the weak $\mathscr{A}$ form on the right-hand side can only be used in a perturbative calculation. Our form[13] (on the left-hand side) can be used nonperturbatively when the effect of this term on the wave function is not small. The logarithmic terms appearing in our Pauli forms provide a natural smoothing mechanism, avoiding the necessity for extra singularity-softening in phenomenological applications.[14]

We have performed numerical calculations of the $q\bar{q}$ bound states using Eq. (2). To solve for the eigenvalue b^2 we use an iterative technique since the Φ's depend on w. We use a spectroscopic notation that describes the quantum numbers associated with the upper-upper decoupled four-component Schrödinger-like equation. The single parameter Λ in Richardson's potential is not fitted to any particular meson or meson family; instead the fit is weighted in favor of mesons with the best known masses. Mesons left out of this fit were the η, η', and others that require annihilation contributions. This "best" fit to the overall meson spectrum gives $\Lambda=0.401$ GeV and b, c, s, and u quark masses of 4.922, 1.574, 0.364, and 0.186 GeV respectively. From Table I (units are in gigaelectronvolts) we see

TABLE I. Meson mass fit produced by one-parameter potential.

Name	Expt.[a]	Theory	Name	Expt.[a]	Theory	Name	Expt.[a]	Theory
Υ: $b\bar{b}$ 1^3S_1	9.460	9.488	ψ: $c\bar{c}$ 1^3D_1	3.770	3.768	ρ: $u\bar{u}$ 1^3S_1	0.770	0.677
Υ: $b\bar{b}$ 2^3S_2	10.021	10.016	ψ: $c\bar{c}$ 2^3D_1	4.159	4.147	ρ': $u\bar{u}$ 2^3S_1	1.620	1.555
Υ: $b\bar{b}$ 3^3S_1	10.351	10.345	ψ: $c\bar{c}$ 3^3D_1	4.415	4.485	δ: $u\bar{u}$ 1^3P_0	0.983	0.783
Υ: $b\bar{b}$ 4^3S_1	10.572	10.607	η_c: $c\bar{c}$ 1^1S_0	2.980	3.013	A_1: $u\bar{u}$ 1^3P_1	1.275	1.189
Υ: $b\bar{b}$ 1^3P_0	9.872	9.855	η_c: $c\bar{c}$ 2^1S_0	3.590	3.598	A_2: $u\bar{u}$ 1^3P_2	1.320	1.334
Υ: $b\bar{b}$ 1^3P_1	9.893	9.890	ψ: $c\bar{c}$ 1^1P_1	3.455?	3.497	g: $u\bar{u}$ 1^3D_3	1.690	1.795
Υ: $b\bar{b}$ 1^3P_2	9.913	9.916	ϕ: $s\bar{s}$ 1^3S_1	1.020	0.972	π: $u\bar{u}$ 1^1S_0	0.139	0.274
Υ: $b\bar{b}$ 2^3P_0	10.234	10.216	ϕ': $s\bar{s}$ 2^3S_1	1.684?	1.715	π': $u\bar{u}$ 2^1S_0	1.300?	1.342
Υ: $b\bar{b}$ 2^3P_1	10.251	10.241	S^*: $s\bar{s}$ 1^3P_0	0.975	1.127	B: $u\bar{u}$ 1^1P_1	1.228	1.185
Υ: $b\bar{b}$ 2^3P_2	10.267	10.259	E: $s\bar{s}$ 1^3P_1	1.420	1.394	A_3: $u\bar{u}$ 1^1D_2	1.680	1.714
B: $b\bar{u}$ 1^1S_0	5.268	5.209	f': $s\bar{s}$ 1^3P_2	1.515	1.521	D^*: $c\bar{u}$ 1^3S_1	2.007	1.949
ψ: $c\bar{c}$ 1^3S_1	3.097	3.123	F^*: $c\bar{s}$ 1^3S_1	2.140	2.085	D: $c\bar{u}$ 1^1S_0	1.863	1.818
ψ: $c\bar{c}$ 2^3S_1	3.686	3.664	F: $c\bar{s}$ 1^1S_0	1.970	1.959	K^*: $s\bar{u}$ 1^3S_1	0.892	0.824
ψ: $c\bar{c}$ 3^3S_1	4.029	4.076				Q_1: $s\bar{u}$ 1^3P_1	1.280	1.284
χ: $c\bar{c}$ 1^3P_0	3.415	3.400				K^*: $s\bar{u}$ 1^3P_0	1.423	1.421
χ: $c\bar{c}$ 1^3P_1	3.510	3.482				K: $s\bar{u}$ 1^1S_0	0.496	0.515
χ: $c\bar{c}$ 1^3P_2	3.556	3.535				Q: $s\bar{u}$ 1^1P_1	1.280	1.297

[a]Ref. 15.

that the π-ρ splitting is 228 MeV off. Nevertheless it is our belief that the pion is not "too relativistic" for the constraint approach.[16] The K-K^* splitting is 87 MeV off. The F-F^*, D-D^*, and ψ-η_c splittings are progressively more acceptable. Note that just as for the experimental results, the ratio $R = (^3P_2 - ^3P_1)/(^3P_1 - ^3P_0)$ predicted for the 1^3P multiplets monotonically decreases with increasing meson mass even though virtually all of the spin dependence in this model comes from the $(1/r)$-like part of the potential. This nonperturbative result contrasts with those of perturbative treatments of Coulomb-like potentials.[17] They predict $R \sim 0.8$ and are rather insensitive to the mass scale. Our ratios are 0.74, 0.65, 0.48, and 0.36 for the $b\bar{b}$, $c\bar{c}$, $s\bar{s}$, and $u\bar{u}$ $1P$ systems versus the experimental values of 0.95, 0.48, 0.21, and 0.15. Our results for R for the heavy mesons compare favorably with those of Ref. 4, that depend on an electrostatically induced long-range Thomas precession. A still better fit for R has been obtained by Schnitzer in Ref. 6 but it uses a three-parameter QCD refinement of the potential. We have also computed the $\langle M_i \rangle$ (the effective constituent masses) for the s and u quarks and found that they are 0.456 and 0.305 GeV, respectively in the ϕ and ρ mesons (consistent with those masses needed to reproduce the simple quark model values for the magnetic moments of the proton and Σ^+ baryon). Thus, if we had fixed $\langle M_u \rangle$ and $\langle M_s \rangle$ initially, we would have had only two free parameters left, m_c and m_b.

Although Richardson's potential is merely an approximation to the static QCD potential and although we have made a crude guess at its covariant generalization, the relativistic wave equation in which we have placed it produces a good overall fit to the meson spectrum with just one potential parameter. In fact, our one-parameter exactly relativistic results are superior to most of those of the multiparameter semirelativistic approaches.[18] The reason for this is that the constraint approach contains more than just relativistic kinematics; it has the capacity to incorporate dynamical recoil effects characteristic of a field theory in a nonperturbative way.

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[1]H. Crater and P. Van Alstine, Ann. Phys. (N.Y.) **148**, 57 (1983), and J. Math. Phys. **23**, 1697 (1982).

[2]P. A. M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University, New York, 1964); I. T. Todorov, Dubna Joint Institute for Nuclear Research Report No. E2-10175, 1976 (unpublished); M. Kalb and P. Van Alstine, Yale University Reports No. C00-3075-146, 1976 and No. C00-3075-156, 1976 (unpublished); P. Van Alstine, Ph.D dissertation, Yale University, 1976 (unpublished); A. Komar, Phys. Rev. D **18**, 1881,1887,3617 (1978); M. King and F. Rohrlich, Phys. Rev. Lett. **44**, 621 (1980); and references contained in Ref. 1.

[3]See L. G. Arnold *et al.*, Phys. Rev. C **23**, 1949 (1981); J. R. Shepard *et al.*, Phys. Rev. Lett. **50**, 1443 (1983).

[4]E. Eichten and F. Feinberg, Phys. Rev. Lett. **43**, 1205 (1979); P. Moxhay and J. Rosner, Phys. Rev. D **28**, 1132 (1983).

[5]M. Arafah, R. Bhandari, and B. Ram, Lett. Nuovo Cimento **38**, 305 (1983); R. W. Childers, Phys. Lett. **121B**, 485 (1983); and M. Bander *et al.*, Phys. Rev. D **29**, 2038 (1984).

[6]Bander *et al.*, Ref. 5; H. J. Schnitzer, "Systematics of Spin-Dependent Quarkonium Forces" (to be published).

[7]J. Richardson, Phys. Lett. **82B**, 272 (1979).

[8]H. Crater and P. Van Alstine, Phys. Lett. **100B**, 16 (1981).

[9]J. Kogut and G. Parisi, Phys. Rev. Lett. **47**, 1089 (1981).

[10]H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One and Two Electron Atoms* (Springer, Berlin, 1957).

[11]B. L. Aneva, J. I. Karchev, and V. A. Rizov, Bulg. J. Phys. **2**, 409 (1975); V. A. Rizov, I. T. Todorov, and B. L. Aneva, Nucl. Phys. B **98**, 447 (1975).

[12]H. Crater and P. Van Alstine, Phys. Rev. D (to be published).

[13]This term is only one of several in $\Phi_{\mathrm{S.S.}}$ which conspire to produce a nonsingular interaction structure. If, as is done by D. B. Lichtenberg *et al.*, Z. Phys. C **19**, 19 (1983), this term is used alone, the interaction it produces for the singlet state is singular unless a smoothing parameter is used.

[14]These are the two-body counterparts of those structures in the one-body Dirac equation that are in part responsible for the success reported in Ref. 3.

[15]M. Aguilar-Benitez *et al.*, Phys. Lett. **111B**, 1 (1982); K. Han *et al.*, Phys. Rev. Lett. **49**, 1612 (1982).

[16]For an interpretation of the term "too relativistic" as it applies to the Breit equation see Childers, Ref. 5.

[17]H. J. Schnitzer, Phys. Rev. Lett. **35**, 1540 (1975).

[18]Those cases where it is not involve (a) two or more parameters applied solely to the ψ and Υ systems, (b) multiparameter QCD refinements, (c) an extremely large number of parameters and/or fits to selective parts of the spectrum.