Comment on "Quantum Fluctuations and the Lorenz Strange Attractor"

In a recent Letter by Elgin and Sarkar[1] it was claimed that the Lorenz strange attractor survives small quantum fluctuations with a mere change in topology. Their analysis was based on the full quantum mechanical master equation of a single-mode laser (a quantum version of the Lorenz model[2]) which was solved approximately by a factorization scheme for its moments. It was found numerically that for weak quantum fluctuations the moments have chaotic time dependence, while for stronger quantum fluctuations the moments exhibit limit-cycle behavior.

In this Comment, I show by general arguments that the time dependence of the moments found in Ref. 1 is incompatible with the master equation and must therefore arise as an artifact of the factorization scheme employed. The master equation requires that the moments in the steady state are time independent. Furthermore, I argue that the fine structure of the Lorenz attractor is incompatible with the uncertainty relations. Therefore, the Lorenz attractor must be washed out at least on a small scale and cannot survive quantization.

The time independence of the moments in the steady state is proven by showing that the density operator which solves the master equation $\dot{\rho} = L\rho$ relaxes to a time-independent density operator $\rho_\infty = \lim_{t\to\infty} e^{Lt}\rho(0)$. That this relaxation indeed takes place follows from the form of L [cf. Eqs. (1)–(5) of Ref. 1], which, even after including the pumping term, may be written in the general form

$$L\rho = -(i/\hbar)[H_I,\rho] + \sum_\lambda([\Gamma_\lambda\rho,\Gamma_\lambda^+] + [\Gamma_\lambda,\rho\Gamma_\lambda^+])$$

Here H_I and the operators Γ_λ do not commute with a common nontrivial observable (except the total number N of atoms, which remains fixed and defines the dimension 2^N of the Hilbert space for the atomic operators). For master equations with generators L with these properties a theorem has been proven by Frigerio[3] (cf. also Spohn[4] and Spohn and Lebowitz[5]), which ensures the relaxation of the density operator and its moments to a unique time-independent steady state. Approximate momentum equations like the ones considered in Ref. 1 which lead to limit-cycle behavior or even chaotic behavior for infinite time therefore miss an essential part of the mixing dynamics described by the master equation.

On physical grounds one would also not expect that the Lorenz attractor could survive quantization as a strange attractor. The reason is that the Lorenz attractor has structure on an arbitrarily fine scale in a three-dimensional phase space, which is spanned by observables which do not commute in the quantized model.[6] Hence if, e.g., the Wigner function is used to represent the density operator in this phase space, this quasi probability density cannot have support on a strange attractor but must be delocalized on a fine scale, as a result of quantum effects. Elsewhere[7] I have presented an exactly solvable quantum model which, in the classical limit, has a strange attractor sitting in a phase space spanned by complementary variables. This model allows one to exhibit the delocalization of the Wigner function due explicitly to quantum effects.

In conclusion, I have presented mathematical and physical arguments that the Lorenz attractor cannot survive quantization. It may be worthwhile to add that analogous arguments can be made to show that the Lorenz attractor cannot survive the addition of classical noise, either. Again, this may be checked by looking at exactly solvable models.[8]

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