Frustrated Instabilities in Nonlinear Optical Resonators

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Instabilities in a nonlinear Fabry-Perot resonator and in an equivalent nonlinear ring resonator with two component cavities are studied. A competition between the time-delayed feedbacks causes a "frustration" in selecting an oscillation mode. The oscillation frequency jumps discontinuously and at random as the ratio of delay times is varied. A number-theoretic method has been successfully applied to elucidate the characteristics of oscillation.

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Chaotic instabilities in nonlinear optical resonators have attracted much attention in the last few years. Various phenomena such as period doubling,[1] destruction of torus,[2] chaotic solitons,[3] and so on[4] have been predicted, and some of them have been observed in experiments.[5]

In the present paper we demonstrate the possibility of an anomalous instability in a nonlinear Fabry-Perot (FP) resonator and in an equivalent ring resonator. These systems each have two time-delayed feedback mechanisms. Competition between these feedbacks cause the systems to fall into a state of "frustration," in which there exist many potential oscillating modes with subtly different stabilities, and a slight change in the cavity length (or delay times) enables an oscillation in a quite different mode. This mechanism is similar to that of the frustration phenomena in thermal equilibrium systems,[6] in which competing interactions of order parameter bring about a complicated number-theoretic order in phase transitions. Number theory is a powerful tool in the analysis of many physical phenomena[7] including frustration.[6] One of the aims of the present paper is to show that elementary number theory is applicable also in laser physics.

The dynamics of a nonlinear FP shown in Fig. 1(a) is described by the Maxwell-Debye equations in the medium[8]:

$$(\pm\partial/\partial z + c^{-1}\partial/\partial t)E^{\pm} = (-\alpha/2 + in)E^{\pm}, \qquad (1a)$$

$$\gamma^{-1}\partial n/\partial t = -(n-n_0) + g(|E^+|^2 + |E^-|^2). \qquad (1b)$$

Here $E^{\pm}(t,z)$ are the electric fields at position z and time t which propagate with light velocity c in the two opposite directions. These have been defined to be dimensionless by $E^{\pm} = \{k|n_2|(1-e^{-\alpha l})/\alpha\}^{1/2}\hat{E}^{\pm}$, where α is the absorption coefficient, n_2 the quadratic coefficient of the nonlinear refractive index, and k the wave number. In Eq. (1b), γ is the relaxation rate of the nonlinear refractive index $n(t,z)$, and $g = (n_2/|n_2|)\alpha(1-e^{-\alpha l})^{-1}$. In Eqs. (1) we have neglected the interaction between E^+ and E^- due to the phase grating, assuming that it diffuses quickly.[8] Equation (1a) together with Eq. (1b) are integrated with respect to z under the boundary conditions $E^+(t,-l_1) = R^{1/2}E^-(t,-l_1) + A$ and $E^-(t,l+l_2) = R^{1/2}E^+(t,l+l_2)$ at the two mirrors, where $A = \hat{E}_I(1-R)^{1/2}\{k|n_2|(1-e^{-\alpha l})/\alpha\}^{1/2}$ is the amplitude of incident laser light. These finally reduce to a set of equations of motion for three variables; $E(t) \equiv E^+(t,0)$, the phase shift ϕ of the electric field across the medium, and an integral of the photon density ψ in the medium,[9]

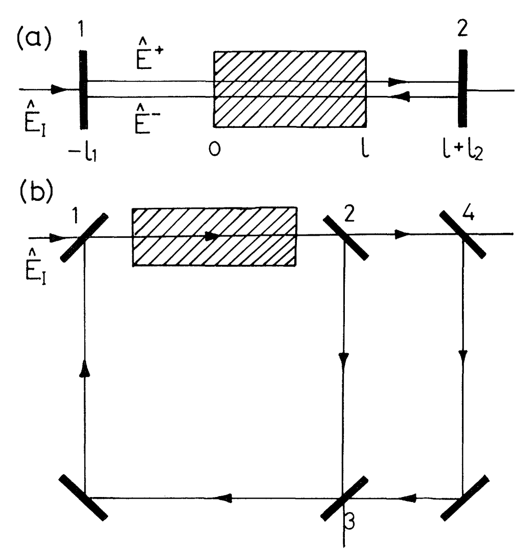

FIG. 1. (a) Fabry-Perot resonator involving a nonlinear dielectric medium, and (b) an equivalent nonlinear ring resonator with two component cavities. Mirrors 1–4 are semitransparent with reflectivities R (1–3) and R' (4).

which are defined by

$$\phi(t)=g\int_0^l dz'\{n(t+(z(z'-l_1)/c,z')+n(t-t_R+(l_1-z')/c,z')\},$$

$$\psi(t)=g\int_0^l dz'\{Re^{\alpha(z'-2l)}|E(t-t_R+2z'/c)|^2+e^{-\alpha z'}|E(t-t_R-2z'/c)|^2\}.$$

These equations involve multiple time delays; one is the usual round-trip time delay $t_1=2(l+l_1+l_2)/c$,[1-3,5] and the others are due to the interaction between the forward- and backward-propagating fields via the medium. A further simplification of these equations is possible in a similar manner as for the ring resonator[1] in the limit of large dissipation, i.e., $B\equiv Re^{-\alpha l}<<1$ with the ratio of the injected energy to dissipation A^2B being kept fixed. In this limit only the shortest additional delay time $t_2\equiv 2l_1/c$ remains, and the reduced equations of motion are

$$\gamma^{-1}\dot{\phi}=-\phi+\psi+|E(t-t_1)|^2, \tag{2a}$$

$$\delta^{-1}\dot{\psi}=-\psi+|E(t-t_2)|^2, \tag{2b}$$

$$|E(t)|^2=A^2\{1+2B\cos[\phi(t)-\phi_0]\}, \tag{2c}$$

where $\delta\equiv\alpha c/2$. To extract the essential properties we assume that the decay of ψ is fast enough to follow the motion of $|E|^2$ adiabatically.[10] Then Eqs. (2) reduce to

$$\gamma^{-}\phi=-\phi+|E(t-t_1)|^2+|E(t-t_2)|^2. \tag{3}$$

If one of the two $|E(t-t_i)|^2$ is discarded, Eq. (3) together with Eq. (2c) is just the delay-differential equation for a single-cavity nonlinear ring resonator.[1] Let us consider a ring resonator with two component cavities as in Fig. 1(b). In this system two feedback loops provide the time-delayed feedbacks corresponding to $|E(t-t_1)|^2$ and to $|E(t-t_2)|^2$, so that its motion is *exactly* described by Eq. (3), if the reflectivities of the mirrors are adjusted so that the feedback strengths are equal.

For a long delay time the time-delayed feedback in general causes the stationary state to be unstable above a threshold value of the incident laser intensity A^2.[1-3,5,8] We investigate the stability of the stationary solution of Eq. (3) for long delay times, i.e., $\tau\equiv\epsilon^{-1}=(t_1+t_2)\gamma>>1$. Linearizing Eq. (3) around the stationary solution ϕ_s and assuming the time dependence $e^{\lambda t}$ for the fluctuation, we obtain an equation for the eigenfrequencies λ. We find that the eigenmodes of the fluctuation have frequencies $\Omega\gamma=\mathrm{Im}\lambda_q=(q\pi-\delta_q)2\gamma/\tau$ specified by positive integers q, where $\delta_q\approx q\pi/\tau$. As the control parameter A is increased, the mode $q=q^*$ which first becomes unstable and thus determines the frequency of self-oscillation is the one which maximizes the function

$$\Psi(q)=(-)^{q+1}\cos(\omega q\pi)\cos\delta_q. \tag{4}$$

Here $\omega\equiv|t_1-t_2|/|t_1+t_2|$ is a ratio of the delay times (or the lengths of the cavities) which plays an important role in determining q^*.

Ψ can be thought of as a net gain function of the mode q which is the product of two competing functions; $\Psi_1\equiv(-)^q\cos(\omega q\pi)$ is the measure of the degree of resonance of the mode q with each of the two component cavities, and $\Psi_2\equiv\cos\delta_q$ is the gain which is a monotonically decreasing function of q. The closer Ψ_1 is to 1, the better the resonance is and the smaller the loss of the mode is. If the ratio of the lengths of the two cavities is rational, i.e., $\omega=P/Q$ (P = even/odd, Q = odd/even), the mode $q=Q$ resonates best with the two cavities. Determination of the mode closest to resonance is difficult when ω is irrational: Rational values of P/Q approximating an irrational ω are dense in its vicinity. The larger the integer Q the better the approximation and the closer the mode $q=Q$ is to resonance. On the other hand, the larger q is, the smaller the gain $\Psi_2\equiv\cos\delta_q$. Consequently the optimum mode q^* is determined by the balance between Ψ_1 and Ψ_2, i.e., the resonance with the two component cavities and the gain.

To find the optimum mode q^* for a given ω, a number-theoretic consideration is necessary. For simplicity, we hereafter consider the problem of maximizing $|\Psi(q)|$ instead of $\Psi(q)$. This gives correct results for $\frac{2}{3}$ of ω in $0<\omega<1$ and does not significantly alter the essential features of the problem, as will be reported elsewhere.[11] Let us consider the continued-fraction expansion (CFE) for ω:

$$\omega=\cfrac{1}{a_1+\cfrac{1}{a_2+\ldots+\omega_n}}\equiv(a_1,a_2,\ldots,\omega_n^{-1}), \tag{5}$$

where $a_i\equiv a_i(\omega)$ are positive integers and ω_n $(0<\omega_n<1)$ is the residual part. Among the rational numbers whose denominators do not exceed $q_n\equiv q_n(\omega)$, the rational number best approximating ω is $p_n(\omega)/q_n(\omega)=(a_1,\ldots,a_{n-1})$ (Lagrange's theorem).[12] Therefore it is quite natural to expect that the optimum q^* is one of the q_n. Indeed it is found that the optimum q^* which maximizes $|\Psi(q)|$ is the q_n which maximizes $|\Psi(q_n)|$. So q_n can be found by minimizing $1-|\Psi_n|$, which for $\tau \equiv \epsilon^{-1} >> 1$ means minimizing

$$\Delta\Psi_n = [\epsilon^{1/2} q_n(\omega)]^2 + [\epsilon^{1/2} q_{n+1}(\omega)]^{-2}. \tag{6}$$

The first term comes from $1-\cos\delta_{q_n} \approx \delta_{q_n}^2/2$, and the second term which is nonzero because of the irrationality of ω comes from $1-|\cos(\omega q_n \pi)| \propto (p_n - \omega q_n)^2$ being well aproximated by q_{n+1}^{-2}.

Next we discuss briefly the rule for generating $q_n(\omega)$. It is evident from Eq. (5) that $a_n(\omega)$ is determined successively by the continued-fraction transformation (CFT):

$$\omega_n = \omega_{n-1}^{-1} - [\omega_{n-1}^{-1}], \quad a_n(\omega) = [\omega_{n-1}^{-1}], \tag{7}$$

with $\omega_1 = \omega$, where $[x]$ denotes the integral part of x. $q_n(\omega)$ is determined by the recursion relation[12]

$$q_n(\omega) = a_n(\omega) q_{n-1}(\omega) + q_{n-2}(\omega). \tag{8}$$

with $q_0(\omega) = 0$ and $q_1(\omega) = 1$. The CFT exhibits strong mixing (and so ergodic) behavior since $|d\omega_n/d\omega_{n-1}| > 1$, in other words, the sequence $\{a_n(\omega)\}$ is chaotic. Therefore, $q_n(\omega)$, which is "driven" by the "random force" $a_n(\omega)$, is also a stochastic variable sensitively dependent upon ω.

Now we return to the problem of minimizing Eq. (6). Identifying $q_{n+1}(\omega)$ with $q_n(\omega)$ in Eq. (6) we may estimate q^* to be $O(\epsilon^{-1/2})$. $q_{n+1}(\omega)$ $[\approx a_n q_n(\omega)]$ is however a random variable little correlated with $q_n(\omega)$, and the above conjecture is correct only qualitatively. A more complete treatment of the minimization of Eq. (6) can be developed[11] with known results for the statistical properties of the CFE.[12] We do not go into the detail of the theory but only refer to the most important result: *For more than 90% of ω in $0 < \omega < 1$, the optimum $q^*(\omega)$ is the largest $q_n(\omega)$ which does not exceed $\epsilon^{-1/2}$.* Using this result we discuss the characteristics of oscillation.

Let us consider the CFE for two neighboring $\omega_1 \neq \omega_2$ (ω_1 irrational), i.e., $\omega_i = (a_1(\omega_i), a_2(\omega_i), \ldots)$, and assume $q^*(\omega_1) = q_n(\omega_1)$. If ω_2 is close enough to ω_1, $a_k(\omega_2) = a_k(\omega_1)$ for all $k \leq n$, so that $q^*(\omega_2)$, i.e., the maximum $q_k(\omega_2)$ smaller than $\epsilon^{-1/2}$, should equal the one at ω_1, i.e., $q^*(\omega_1)$. As ω_2 is moved away from ω_1, the $a_k(\omega_2)$ successively shift from the values $a_k(\omega_1)$ in the order $k = n, n-1, \ldots,$ and at some critical value $\omega_2 = \omega_1 + \Delta\omega$, $q^*(\omega_2)$ jumps from $q^*(\omega_1)$ to some other value. The characteristic width of $\Delta\omega$ is estimated by considering that when $q_n(\omega_1)$ exceeds $\epsilon^{-1/2}$ the nth iterate of a small interval $\Delta\omega$ by the CFT is amplified to $O(1)$. This gives $\Delta\omega \sim \epsilon = 2/\tau$. Because of the chaotic behavior of the sequence $\{a_k(\omega_2)\}$, $a_k(\omega_2)$ becomes uncorrelated with $a_k(\omega_1)$ for $k \leq n$ as $|\omega_1 - \omega_2|$ is increased beyond $\Delta\omega$, and so similarly $q^*(\omega_2)$ becomes uncorrelated with $q^*(\omega_1)$. Therefore, the frequency of oscillation Ω exhibits singular behavior in the limit $\tau >> 1$; as ω is varied on a scale of τ^{-1}, Ω changes stepwise, and moreover the height of steps is a random function of ω on a scale $>> \tau^{-1}$. The statistical distribution of Ω can be evaluated by use of the statistical properties of the CFT.[11] It is given by

$$P(\Omega) \propto \theta(1-\Omega/\epsilon^{1/2}\pi)\ln(1+\Omega/\epsilon^{1/2}\pi)/\Omega, \tag{9}$$

where $\theta(x)$ is the Heaviside step function. Hence the average frequency of oscillation is proportional to (delay time)$^{-1/2}$, i.e., $\Omega_{av} = \{12\sqrt{2}(2\ln 2 - 1)/\pi\}[\gamma(t_1+t_2)]^{-1/2}$, which differs greatly from the usual dependence, i.e., (delay time)$^{-1}$.[1,5,8] Figures 2(a)–2(c) show the oscillation frequency Ω as a function of ω, which is obtained numerically by maximizing the exact gain function $\Psi(q)$. As is expected, $\Omega(\omega)$ becomes complicated as τ is increased, and in Fig. 2(c) $\Omega(\omega)$ looks as if it is quite random. Figure 2(d) is an enlargement of Fig.

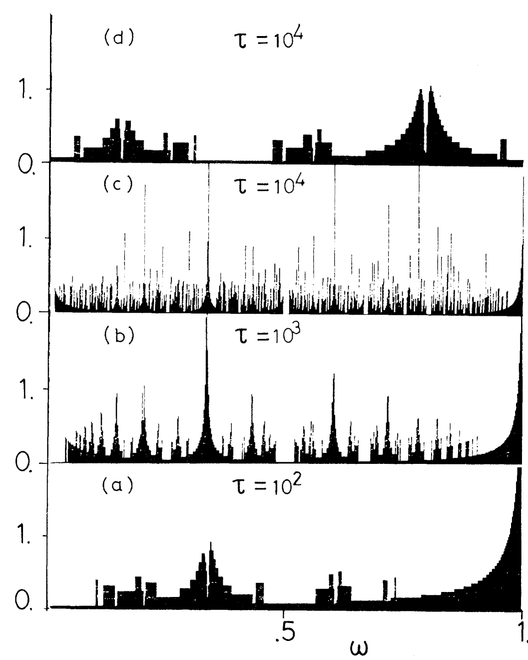

FIG. 2. Oscillation frequency $\Omega/2\pi\sqrt{\epsilon}$ as a function of ω for various values of τ. (d) is an enlargement of (c) for $0.15 < \omega < 0.16$.

2(c), which indicates that the random variation on a coarse scale consists of uneven steps on a fine scale.

In conclusion, we have shown frustration phenomena in the instabilities exhibited by models of a nonlinear Fabry-Perot resonator and of an equivalent nonlinear ring resonator. The competition between multiple time delays causes frustration in the selection of the most unstable mode. Such phenomena may be intrinsic in unstable optical systems with compound cavities.[13] We note that there remains unsolved problems within the framework of linear stability analysis: Increasing A (for a fixed ω), we observed the singular oscillation behavior seen in Fig. 2. Is the same behavior observed as ω is changed for fixed A? Are the discontinuous jumps accompained by hysteresis, or not? To answer these questions experimental verification is strongly desired. An experiment would be more easily performed in a hybrid bistable device with two time-delayed feedbacks.

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