Scaling Behavior of Self-Avoiding Random Surfaces

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Self-avoiding random surfaces are analyzed by renormalization-group methods. The Hausdorff dimension $1/\nu$ and the critical plaquette fugacity are computed for different dimensionalities d; in particular, $\nu=\frac{1}{2}-\epsilon/4+O(\epsilon^2)$ for $d=2+\epsilon$. The model describes "sheet polymers" in a good solvent: A Flory type of argument yields $\nu=3/(4+d)$, in good agreement with the renormalization results, and a critical dimensionality $d_c=8$, with $\nu=\frac{1}{4}$.

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Models of self-avoiding random surfaces (SAS's) are natural generalizations of the self-avoiding walk (SAW) problem.[1] In the context of lattice gauge theories SAS's are expected to play a role analogous to the one of SAW's for spin systems,[2] and a study of their scaling properties should lead to a better understanding of these theories and their continuum limits.

Besides representing an interesting problem in lattice statistics on their own, SAS's can be viewed as realistic generalizations of the surface considered, e.g., in the solid-on-solid model,[3] which describes solid surfaces and interfaces. There is nowadays clear experimental evidence that the surfaces of many materials present irregularities with remarkable self-similarity over several length scales in the molecular range.[4] The Hausdorff fractal dimensions of these surfaces are actually measured by monolayer coverage techniques.[4] We thus hope that the theoretical study of models like SAS's and of their fractal properties will be of interest for surface physics.

As emphasized by the present approach, SAS's can also be expected to describe configurational properties of single flexible two-dimensional "sheet polymers"[5] with excluded-volume effects.

By SAS's we mean here a collection of elementary plaquettes on a d-dimensional hypercubic lattice. The plaquettes should be glued together in such a way as to form a connected aggregate, without overlap, in which each lattice link belongs to, at most, two plaquettes.[6] Furthermore, the surfaces are orientable, without handles, and either have a contour (given by the bonds belonging to a single plaquette), or have the topology of a sphere. The set of all SAS's with a given fixed contour represents, in a gauge theory, typical strong coupling contributions to the Wegner-Wilson loop correlation function, which defines the string tension.[2] In the context of our model the role of the Wegner-Wilson loop is played by a properly defined generating function of the general type defined below.

Like for SAW's, we are primarily interested in discussing the asymptotic behavior of a characteristic length, ξ, of the surfaces, when the total number of plaquettes, N, approaches infinity. We can expect that ξ, e.g., the average radius of gyration with respect to the "center of mass," behaves like $\xi\propto N^{\nu}$, for large N, ν being a proper exponent, whose reciprocal simply gives the Hausdorff dimension of the critical SAS's.[7] Instead of working with all surfaces having a fixed number, N, of plaquettes, the problem is most conveniently formulated in a grand-canonical context, where a fugacity K is associated to each plaquette; the grand-partition function (generating function) is just a sum over distinct surfaces S, of a given topology and containing a given point, of $K^{|S|}$, $|S|$ being the area of S. The grand-canonical average radius of gyration is then a function of K, defined for K lower than some critical value, K_c, and diverging like $(K_c-K)^{-\nu}$ for $K\rightarrow K_c^-$.

Our strategy for renormalizing SAS's consists of finding a regular mapping $K'=K'(K)$, such that

$$\xi(K')=l^{-1}\xi(K), \tag{1}$$

l being a length rescaling. As a result of the assumed regularity of $K'(K)$, the singular behavior of ξ at K_c implies $K'(K_c)=K_c$ and $dK'/dK\,|_{K_c}=l^{1/\nu}$.

Our recipe for constructing an approximate mapping for SAS's has strong analogies with similar work on the SAW problem,[8] and can be best illustrated in a three-dimensional example. In order to obtain a rescaling $l=2$, we partition the lattice into cubic cells of side 2; the plaquettes on the boundary common to two adjacent cells are assumed to belong to the one with higher values of the coordinate normal to the boundary.

In a given cell, all simply connected surfaces realizing an effective full covering parallel to a given face are then assumed to contribute, at a coarse grained level, to the fugacity K' of the correspond-

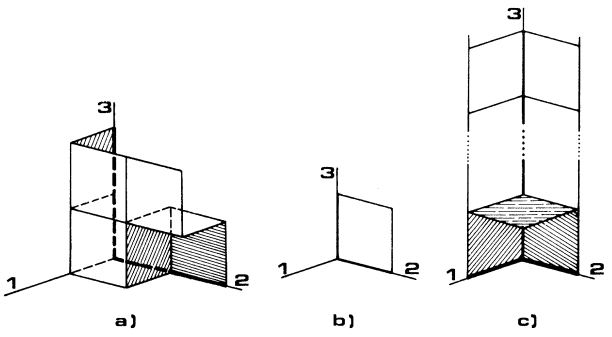

FIG. 1. Surface with nine plaquettes contributing to K' in Eq. (2). (a) The heavy lines, continuous and dashed, show the fixed part of the border. (b) The corresponding renormalized plaquette is in the (2,3) plane. In (c) the anisotropic cell is shown and a particular (three-plaquette) surface contributing to K' in the (1,2) plane is reported.

ing effective plaquettes of a rescaled elementary cell [Figs. 1(a) and 1(b)]. Like in the "corner rule" for SAW,[8] in order to avoid overcounting, one must consider surfaces with part of the border fixed [e.g., coinciding with two consective sides of the cell face to which the renormalization refers, see Fig. 1(a)]. In this way just fourteen surfaces contribute to the mapping, giving

$$K' = K^4 + K^6 + 4K^7 + 3K^8 + 4K^9 + K^{10}, \qquad (2)$$

which has a nontrivial fixed point $K_c = 0.651$ with $\nu = 0.37$. At present these results can only be compared with the Monte Carlo estimate $K_c = 0.588$ for spherical SAS in $d = 3$.[9]

Unlike the case of SAW's, extension of the above scheme of calculation to higher values of d and/or l for SAS's would pose formidable enumeration problems. However, we could obtain transformations with explicit parametric dependences on l and d, by performing successive anisotropic rescalings in each single lattice direction.[10]

In order to rescale the system by a factor l along one of the coordinate directions, we use a cell as illustrated in Fig. 1(c) for $d = 3$. Fugacities, $K'_\perp$, of plaquettes perpendicular to the rescaling direction are then given by

$$K'_\perp = R(K) = K(K^{2l} - 1)/(K^2 - 1), \qquad (3a)$$

whereas fugacities $K'_\parallel$ of plaquettes parallel to that direction are simply

$$K'_\parallel = K^l. \qquad (3b)$$

Successive application of d different anisotropic recursions like (3a) and (3b) yields the result of a global rescaling l. At this stage we make the approximation of forgetting the anisotropy generated in this way and choose as renormalized K' the result of applying (3a) and (3b) $d-2$ and 2 times, respectively, on K. The order in which the transformations are successively applied does not sensibly affect the final results. A similar anisotropic scheme applied to SAW's gives results which, in all d, compare very well in quality with those of more involved isotropic renormalization schemes.[11] As a rule, for SAW's it turns out that the best results are obtained when the transformations like (3a) are performed first; so we made the same choice, which gives $K' = [R^{d-2}(K)]^{l^2}$ for SAS's. In Table I several values of K_c and ν obtained by this method for different dimensionalities are reported. At this stage, unfortunately, we cannot compare the results of Table I with other estimates. However, it is important to notice that the similar scheme applied to the SAW's [11] gives rather satisfactory results: e.g., for $l = 3$ and $d = 2$, we get $K_c = 0.408$ and $\nu = 0.74$ (expected $K_c = 0.380$ and $\nu = 0.75$[8]), and for $l = 3$ and $d = 3$, $K_c = 0.243$ and $\nu = 0.63$ (expected $K_c = 0.213$ and $\nu = 0.59$[8]). Furthermore, in the SAW case, for $l = 1 + \delta$ and $d = 1 + \epsilon$ (δ and ϵ small and positive), we obtained by the same scheme $K_c = 1 - \epsilon + O(\epsilon^2)$ and $\nu = 1 - \epsilon/2 + O(\epsilon^2)$. These last results were obtained previously in a different way, and conjectured to be exact.[8,12]

For SAS's, by choosing l infinitesimally close to

TABLE. I. In the first and second row values of K_c and ν obtained by our renormalization scheme are reported. In the third row are the values of ν from Eq. (7).

l \ d	3		4		5		6		7		8	
	K_c	ν	K_c	ν	K_c	ν	K_c	ν	K_c	ν	K_c	ν
2	0.636	0.38	0.491	0.32	0.412	0.29	0.360	0.27	0.323	0.26	0.295	0.24
3	0.619	0.39	0.477	0.34	0.400	0.31	0.350	0.29	0.315	0.28	0.288	0.27
	0.588[a]	0.43		0.37		0.33		0.30		0.27		0.25

[a]Ref. 9.

1, we get

$$K' = K + \frac{2K \ln K}{1-K^2}[1-(d-1)K^2]\delta + O(\delta^2), \tag{4}$$

which gives $K_c = 1/(d-1)^{1/2}$ and $\nu = (d-2)/2(d-1)\ln(d-1)$. These results are exact for $d=2$, where we obviously must have $K_c = 1$ and $\nu = \frac{1}{2}$. Like in the case of SAW's, we can expect Eq. (4) to be a good approximation for d low and close to 2. For $d = 2+\epsilon$, in particular, we get $K_c = 1 - \frac{1}{2}\epsilon + O(\epsilon^2)$ and $\nu = \frac{1}{2} - \epsilon/4 + O(\epsilon^2)$ for SAS's.

Unlike the lower critical dimensionality (2 in our case), the upper critical dimensionality, d_c, is not easy to determine with real-space renormalization methods. By applying to our surfaces methods similar to those used by Parisi and co-workers[13] for gauge theories, one can obtain $K_c \sim 0.45 \times (d-2)^{-1/4}$ and $\nu \to \frac{1}{4}$ for $d \to \infty$.[11] The last results are derived by taking into account the surfaces giving the dominant contribution to the generating function for $d \to \infty$; these results can also be interpreted as a "classical" approximation for the problem. Let us remark that this "classical" K_c is very close to our renormalization-group estimates for the highest values of d considered.

When the self-avoiding constraint is released our SAS's become equivalent to planar random surfaces,[6] for which $\nu = \frac{1}{4}$ turns out to be compatible with Monte Carlo results in $d=2$ and 3 [14] and (for $d \to \infty$) with the results of Durhuus, Fröhlich, and Jonsson.[15,16]

Thus we conclude that $\nu = \frac{1}{4}$ qualifies as the most plausible "classical" value for SAS's. Such a classical value of course should imply $d_c = 8$, just as for SAW's $\nu = \frac{1}{2}$ implies $d_c = 4$.

The connection between SAS's and "sheet polymers" suggests that we set up a Flory approximation for the ν exponent.[1,5] We imagine that our surfaces represent configurations of two-dimensional polymers diluted in a good solvent. The average radius of gyration of such polymers is thus expected to be determined by two competing effects. The first is a repulsive one, tending to "swell" the polymer, and is determined by short-range two-particle encounters. In a dilute solution the corresponding repulsive mean-field free energy is thus estimated as

$$F_{\text{rep}} \sim N^2/\xi^d. \tag{5}$$

The latter effect is an elastic one, tending to reduce ξ to its value in the absence of excluded-volume effects. For linear polymers (SAW) the elastic free energy is just approximated by the one appropriate to a Gaussian chain, $F_{\text{el}} \sim \xi^2/N$. For random surfaces it is natural to expect the area (e.g., of the projection of the surface on a lattice plane) to be Gaussian distributed, instead of the radius of gyration. This is also consistent with the expected $\nu = \frac{1}{4}$ for random surfaces since such an area has to be proportional to ξ^2, and $\xi \propto N^{1/4}$. These considerations suggest a term

$$F_{\text{el}} \sim \xi^4/N, \tag{6}$$

for our SAS's. It should be noticed that the guess (6), even if very plausible, cannot be justified by a direct enumeration of purely random surface configurations. Unlike for random walks, such enumeration is beyond our present theoretical capabilities. By applications of standard arguments[1,5] Eqs. (5) and (6) lead to

$$\nu = 3/(4+d), \quad 2 \leq d \leq 8, \qquad \nu = \tfrac{1}{4}, \quad d > 8. \tag{7}$$

This Flory-type formula is consistent with $d_c = 8$ if the classical value of ν is $\frac{1}{4}$; moreover, it has the nice feature of correctly predicting the value $\nu = \frac{1}{2}$ for $d = 2$.

For SAW's and other similar problems, the Flory approach, besides predicting the correct upper and lower critical dimensionalities, gives a very reasonable approximation for ν. Comparison of the values of ν given by (7) (last row in Table I) with those obtained independently by our renormalization approach is thus a good test. As shown in Table I, the degree of consistency is of the same standard as in the case of SAW's or other lattice problems, to which Flory approximations and real-space methods have been applied.[8] It is worthwhile noticing that the ν values obtained by our renormalization method turn out to be quite close to the expected exact value for $d=8$; similar agreements are also obtained by our scheme for the SAW case (e.g., $\nu = 0.54$ in $d=4$).

Unlike SAW's, which can be considered as the $n \to 0$ limit of an n-vector model,[1] SAS's are not a special limit of some gauge model (see Ref. 6 and references therein). As a consequence field theoretical techniques cannot be applied to our model, and approaches like those presented here are the only way, besides direct simulation, of getting information on the critical properties.

For going beyond the level of accuracy obtained here with our techniques, a better control of the hard enumeration problems posed by surfaces in

$d \geq 3$ will be needed in the future.

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