Crossover in an Exactly Solvable Dimer Model of Domain Walls with Dislocations

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An exactly solvable generalized dimer model is proposed. In one limit this model is the isotropic Ising model and in another limit it is the anisotropic Kasteleyn model with radically different critical behavior. The value of the crossover exponent ϕ near the multicritical point is determined to be $\frac{1}{2}$. The proposed model is isomorphic to a domain-wall model with dislocations, thereby providing an example of dislocation-mediated crossover in uniaxial commensurate-incommensurate transitions.

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Consider, for the lattice in Fig. 1(a), the partition function

$$Z(u,v,w)=\sum_{\text{states}} u^p v^q w^r,$$

where each state consists of a complete covering of the lattice by dimers such that each lattice site is connected by a dimer to one and only one of its nearest neighbors. Dimers covering the edges labeled u, v, and w in Fig. 1(a) have different energies ϵ_u, ϵ_v, and ϵ_w such that the activities (Boltzmann factors) are respectively u, v, and w. The numbers of dimers covering the u-, v-, and w-type edges are respectively p, q, and r with $p+q+r=N/2$, where N is the total number of lattice sites. Because of this constraint there are only two parameters in the model and so there is no loss in generality in setting $w=1$ (i.e., $\epsilon_w=0$). One of the remaining parameters is eliminated by specifying the temperature T on the energy scale of the u dimers as $\epsilon_u/kT=-\ln u$ and the final parameter can be thought of as $\epsilon_u-\epsilon_v$ or, equivalently, v/u. This generalized dimer model becomes a specific model as a function of T when v/u is specified. Furthermore, since this is a planar model it can be *solved exactly* by the Pfaffian technique.[1]

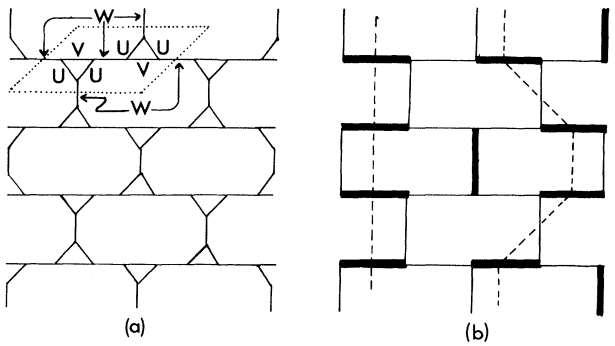

FIG. 1. (a) The lattice for the proposed dimer model where u, v, and w refer to the activities of the edges. Without any loss of generality w can be taken to be 1. A unit cell is shown by the dotted lines. (b) The K model on the brick lattice.. The horizontal edges have activity x and the vertical edges have w which can again be taken to be 1. By connecting the centers of the horizontal dimers (thick lines) one gets the domain walls (dashed lines) with $p=1$. (See text.)

For the case $v/u=1$, the generalized model becomes the nearest-neighbor *isotropic Ising model* on the brick lattice (except for some trivial factors). This is easily seen from the one-one correspondence of the dimer state at each vertex to the high-temperature "hyperbolic tangent" expansion of the partition function of the Ising model[2] (Fig. 2). As is well known,[1] the exact thermodynamic behavior is a *logarithmically diverging specific heat* [$\alpha=O(\log)$] at $3u^2=1$.

Now consider the case $v/u=0$. For this special case, Fig. 2 shows a transformation to a simpler dimer model on the brick lattice shown in Fig. 1(b). The partition function for this latter model is just $Z_K=\sum x^{p/2}$ where $x=u^2$ and the sum is over the allowed states. This model, called *the K model*,[3] was first solved by Kasteleyn[4] who showed that the

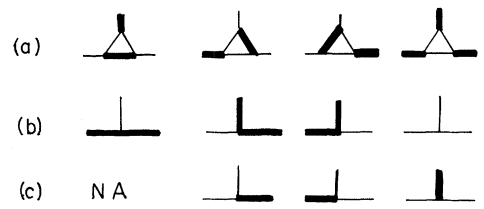

FIG. 2. (a) City configurations for the proposed model. (b) The equivalence, for $v=u$, with the bond configurations on the brick lattice for the Ising partition function. (See Ref. 2.) (c) The correspondence, for $v=0$, with the K model of Fig. 1(b). NA stands for "not allowed."

specific heat is zero for $x < x_c = \frac{1}{2}$ and *diverges with an exponent* $\alpha = \frac{1}{2}$ for $T > T_c$, i.e., $x > x_c$. This behavior, which has been called[3] a $\frac{3}{2}$ order transition, is so radically different from the Ising-like behavior that it is quite natural to think of *these two models as belonging to two different universality classes.* Every model[2,4,5,6] with a phase transition that has been solved exactly by the Pfaffian technique belongs to one or the other of these two classes.

The K model is also of current interest because it is a special case[7,8] of domain-wall models[9] [Fig. 1(b)] of uniaxial $p \times 1$ commensurate-incommensurate (CI) transitions where p is the number of distinguishable phases. Each domain wall must proceed along the vertical direction and is forbidden to end or to meet another wall. In terms of the $p \times 1$ CI classification scheme the K model has been described[7,8] as a $p = 1$ model because the phases on either side of a domain wall are indistinguishable. This phase degeneracy does not reduce the value of the K model in providing critical tests of general theories of $p \times 1$ CI transitions. For instance, the exponent $\bar{\beta}$, which describes the average spacing l between domain walls as the reduced temperature $t[\equiv (T - T_c)/T_c] \to 0$ according to $l \sim t^{-\bar{\beta}}$, is predicted[9] to be $\frac{1}{2}$ in two dimensions *independent* of p. For the K model the internal energy is proportional to $1/l$ and so $\bar{\beta} = \alpha = \frac{1}{2}$. Therefore the general theory of $p \times 1$ CI transition is in accord with the exact result for the K model.

The new generalized model also has a domain-wall analogy as shown in Fig. 3. For nonzero values of v the walls can meet and annihilate in pairs at the centers of the triangles as at P. These points represent the dislocations[9] in the domain-wall model, with activity v. Domain walls of finite lengths (represented by A in Fig. 3) *destroy* the low-temperature frozen behavior of the K model, introducing critical fluctuations[10] as $T \to T_c -$. In order to study the crossover from a model with dislocations to one without, the thermodynamic behavior of the free energy near T_c will now be considered as a function of v.

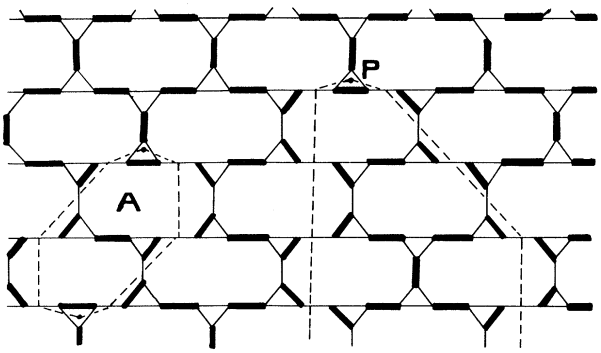

FIG. 3. Domain walls in the new model are obtained by connecting the centers of the vacant horizontal $w(=1)$ type edges. If a v-type dimer is present the walls are said to meet and annihilate each other at the center of the triangle, as at P. Excitations of type A are called localized excitations. Neither these nor the P dislocations are allowed in the K model.

In the thermodynamic limit the free energy f of the new model, derived exactly by the Pfaffian technique, is[1]

$$f = -kTN^{-1}\ln Z = -kT(24\pi^2)^{-1}\int_0^{2\pi}\int_0^{2\pi} \ln[g_0 + 2by + y^2]\,d\theta_1 d\theta_2, \tag{1}$$

where

$$g_0 = 1 + 2x^2 + 2x^2\cos(\theta_1 - \theta_2) - 2x(\cos\theta_1 + \cos\theta_2),$$

$$b = x(\cos\theta_1 + \cos\theta_2) - \cos(\theta_1 - \theta_2), \quad x = u^2, \quad y = v^2.$$

The critical point is determined by the equation $2x + y = 1$ when the argument of the logarithm vanishes for $\theta_1 = \theta_2 = 0$. The phase diagram is shown in Fig. 4. The transition is Kasteleyn type for $y = 0$ and Ising type if $y \neq 0$. Our focus is on the multicritical point[11] at $x = \frac{1}{2}$, $y = 0$ (to be called the *Kasteleyn point*) where v (or y) is a relevant variable[11] because it changes the transition behavior radically.

Near the multicritical point, the free energy f is expected to have a scaling form[11] of the following type:

$$f \approx t^{2-\alpha} W_{\pm}(yt^{-2\phi}) \tag{2}$$

in the limit $v \to 0$ and $t \to 0$. $W_{\pm}$ are the scaling functions for $t \gtrless 0$ and ϕ is the crossover exponent for v. The exponent α is $\frac{1}{2}$ for the K model. Since v is a relevant variable near the Kasteleyn point, the exponent ϕ must be positive.[11] For simplicity we consider only the function W_- since the exponent ϕ is expected to be the same on both sides of T_c.

The strategy to determine ϕ is to expand $W_-(z)$ in powers of z so that, from (2),

$$f \approx \sum_n t^{-2n\phi + 3/2} W_-^{(n)}(0) y^n / n!, \tag{3}$$

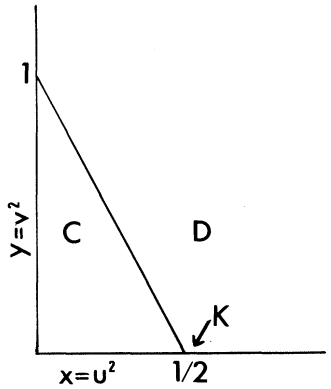

FIG. 4. Phase diagram in the (x,y) plane. The transition is Kasteleyn or commensurate-incommensurate type for $y=0$, for which dislocations are not allowed, and Ising type for $y\neq 0$, for which dislocations are allowed. The point $K=(\frac{1}{2},0)$ is the multicritical point, called the Kasteleyn point. C represents the commensurate phase and D the disordered phase.

where $W_{-}^{(n)}(z)$ is the nth derivative of $W_{-}(z)$ with respect to z. The exponent ϕ can then be found from the leading t dependence of the coefficients of these terms.

To obtain a form of the type (3) from (1), note that the temperature-dependent part of the coefficient of y^n is the double integral over θ_1 and θ_2 of $(\gamma_+^n+\gamma_-^n)/g_0^n$ where $\gamma_\pm$ are the roots of the quadratic equation $y^2+2by+g_0=0$. For $n=1$, this double integral of b/g_0 can be evaluated exactly for $x<\frac{1}{2}$ by transforming each of the integrals into a contour integral. The leading t dependence is found to be $t^{1/2}$ giving $\phi=\frac{1}{2}$. For arbitrary n, the leading t dependence comes from a region close to $\theta_1=\theta_2=0$ where $g_0=0$ for $x=\frac{1}{2}$. Expanding the integrand about the origin in the θ_1, θ_2 plane, one can in fact show that the leading t dependence of the nth term of (3) is $t^{-n+3/2}$. This proves that y/t is the scaling variable in all orders of y.

The shift of the critical temperature also shows that the crossover exponent ϕ is $\frac{1}{2}$. The scaling form (2) requires[11] that the shift t_c in the critical temperature should go as $v^{1/\phi}$ as $v\to 0$. From the exact phase boundary, we find that near the Kasteleyn point, $x\to\frac{1}{2}$ and $v\to 0$, this shift is given by

$$t_c\sim(1-2x)\sim y=v^2$$

which identifies $\phi=\frac{1}{2}$.

The value of the crossover exponent ($\phi=\frac{1}{2}$) is in disagreement with the formula $\phi=(6-p^2)/4$ given by Huse and Fisher[12] if the previously proposed value[7,8] of $p=1$ for the K model is used. This disagreement is more apparent than real because it was assumed by Huse and Fisher that precisely p walls meet at a dislocation point of a $p\times 1$ model. Since the proposed dimer model involves pairwise annihilation of the walls, $p=2$ should be used in the formula of Huse and Fisher. It may be desirable to introduce two symbols, (i) p for the number of distinct phases and (ii) q for the minimum number of walls that can meet at a point, so that the general formula of Huse and Fisher involves q and *not* p. For the K model $q=2$ even though $p=1$ and therefore the exact value of the crossover exponent agrees with the general formula.

The behavior of the correlation functions of the K model is also radically different from that in the Ising model. Indeed, an order parameter related to the correlation function has not been identified[3] for the K model. It will be the object of future work to investigate the crossover of the order parameter and the correlation functions as the Kasteleyn point is approached. Such studies should also determine the critical index η which has been predicted to depend[9] on p.

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[1]C. J. Thompson, *Mathematical Statistical Mechanics* (Princeton Univ. Press, Princeton, N.J., 1972).
[2]M. E. Fisher, J. Math. Phys. **7**, 1776 (1966).
[3]J. F. Nagle, Phys. Rev. Lett. **34**, 1150 (1975).
[4]P. W. Kasteleyn, J. Math. Phys. **4**, 287 (1963).
[5]S. R. Salinas and J. F. Nagle, Phys. Rev. B **9**, 4920 (1974).
[6]J. F. Nagle and G. R. Allen, J. Chem. Phys. **55**, 2708 (1971).
[7]S. M. Bhattacharjee, J. F. Nagle, D. A. Huse, and M. E. Fisher, J. Stat. Phys. **32**, 361 (1983).
[8]M. E. Fisher, J. Stat. Phys. **34**, 667 (1984).
[9]P. Bak, Rep. Prog. Phys. **45**, 587 (1982).
[10]The STS model of Ref. 6, belonging to the same universality class as the K model, also has localized excitations but it does not show any critical fluctuations as $T\to T_c-$.
[11]A. Aharony, in *Critical Phenomena*, edited by F. J. W. Hahne, Lecture Notes in Physics Vol. 186 (Springer-Verlag, New York, 1983), p. 209.
[12]D. A. Huse and M. E. Fisher, Phys. Rev. B **29**, 239 (1984).