Cartan-Frobenius Integration Method and Exact Solutions for Relativistic Ideal Fluid Flows

Gerald Rosen
Department of Physics, Drexel University, Philadelphia, Pennsylvania 19104
(Received 27 December 1983)

It is observed that all solutions to the quasilinear ideal fluid flow equations $[(\rho+c^{-2}p)u_\mu u_\nu]_{,\nu}=-p_{,\mu}$ with $p=p(\rho)$ and $u_\nu u_\nu\equiv -c^2$ can be classified algebraically in terms of three basic types of solutions and combinations thereof. This classification engenders the definition of *symmetric flows,* a broad class for which a general intermediate integral is derived here by application of the Frobenius integration theorem. Representative exact analytical solutions are presented.

PACS numbers: 47.10.+g, 03.40.Gc, 47.75.+f

Nonlinear partial differential equations (NPDE) which admit linear-equivalence mappings[1,2] generally feature only two independent variables (e.g., x and t) and are *quasilinear* in the sense of being linear in the highest-order partial derivatives with respect to the two independent variables; extensions of such mappings for two- and three-dimensional $\vec{x}$ do not result in associated linear equations. Thus, the linear-equivalence solutional method[1] cannot be extended directly for three or four independent variables.

Can one formulate an analytical integration procedure which makes use of the structural aspect of quasilinear NPDE for cases of more than two independent variables? In particular, can the first-order quasilinear homogeneous character of the ideal fluid flow equations, which engenders linearization by the hodograph transformation if and only if the flow is one dimensional,[3] be utilized in an analytical integration method for the cases of two- and three-dimensional time-dependent fluid flows? In this Letter I present an efficient analytical solutional method which does indeed utilize the first-order quasilinear homogeneous character of ideal fluid flow while side-stepping the untractable nonlinearities of a hodograph extension.

Consider a relativistic ideal fluid flow governed by the equations[4,5]

$$[(\rho+c^{-2}p)u_\mu u_\nu]_{,\nu}=-p_{,\mu} \tag{1}$$

in which the proper mass density ρ (including both material and internal energy density of the fluid) and proper pressure p are related by an algebraic equation of state[6] $p=p(\rho)$, the fluid velocity $\vec{v}$ is related to $(u_1,u_2,u_3)\equiv\vec{u}$ by $\vec{v}=(1+c^{-2}\times|\vec{u}|^2)^{-1/2}\vec{u}$ or equivalently $\vec{u}\equiv(1-c^{-2}\times|\vec{v}|^2)^{-1/2}\vec{v}$, the quantity $u_4\equiv i(|\vec{u}|^2+c^2)^{1/2}$ (so that $u\cdot u\equiv u_\nu u_\nu\equiv -c^2$), and subscripts after commas denote differentiation with respect to the space-time coordinates $\vec{x}=(x_1,x_2,x_3)$ and $x_4\equiv ict$. For specialized equations of state (principally $p=\frac{1}{3}c^2\rho$), one-dimensional time-dependent and two-dimensional steady solutions to (1) have been derived by the Riemann, self-similarity, and stream-function methods.[7] The present communication reports exact analytical solutions to (1) for general equations of state $p=p(\rho)$.

One starts by introducing the timelike *flow vector*

$$f_\mu\equiv(\rho+c^{-2}p)^{1/2}u_\mu \tag{2}$$

with four independent components. In terms of (2), Eqs. (1) become

$$f_{\mu,\nu}f_\nu+f_\mu f_{\nu,\nu}=\alpha f_\nu f_{\nu,\mu}, \tag{3}$$

where

$$\alpha=\alpha(f)\equiv 2[(dp/d\rho)+c^2]^{-1}dp/d\rho \tag{4}$$

is a function of the scalar quantity

$$f\equiv(-f\cdot f)^{1/2}\equiv(-f_\nu f_\nu)^{1/2}=(c^2\rho+p)^{1/2} \tag{5}$$

and the relation $f_\nu f_{\nu,\mu}=-ff_{,\mu}$ is employed. Linear homogeneous in the first derivatives of the flow vector, Eqs. (3) can be solved algebraically for $f_{\mu,\nu}$; in the general case one obtains

$$f_{\mu,\nu}=\sum_{i=1}^{3}A_{\mu\nu}^{(i)}, \tag{6}$$

where

$$A_{\mu\nu}^{(1)}=\xi[\delta_{\mu\nu}+(2-\alpha)^{-1}(5-\alpha)f^{-2}f_\mu f_\nu], \tag{7}$$

$$A_{\mu\nu}^{(2)}=f_\mu\eta_\nu+\alpha\eta_\mu f_\nu,\quad \eta_\nu f_\nu\equiv 0, \tag{8}$$

$$A_{\mu\nu}^{(3)}=\zeta_{\mu\nu},\quad \zeta_{\mu\nu}f_\nu\equiv f_\nu\zeta_{\nu\mu}\equiv 0\equiv\zeta_{\nu\nu}. \tag{9}$$

Involving one, three, and eight parameter functions respectively in ξ and the linearly independent components of η_μ and $\zeta_{\mu\nu}$, the tensors (7)–(9) have the trace values and orthogonality properties

$$A_{\nu\nu}^{(1)} = 3\xi(2-\alpha)^{-1}(1-\alpha),$$
$$A_{\nu\nu}^{(2)} = A_{\nu\nu}^{(3)} = 0, \qquad (10)$$

$$A_{\mu\nu}^{(i)}A_{\mu\nu}^{(j)} = A_{\mu\nu}^{(i)}A_{\nu\mu}^{(j)} = 0 \quad \text{for } i \neq j. \qquad (11)$$

The representation (6) for $f_{\mu,\nu}$ satisfying (3) is complete and unique because (10) implies that

$$\xi = \tfrac{1}{3}(2-\alpha)(1-\alpha)^{-1}f_{\nu,\nu}, \qquad (12)$$

while (12) and the contraction of (6) with f_ν produces

$$\eta_\mu = f^{-1}f_{,\mu} - (1-\alpha)^{-1}f^{-2}f_\mu f_{\nu,\nu}. \qquad (13)$$

Hence (7) and (8) are unique correspondents of a flow vector field f_μ, and (9) then follows as $A_{\mu\nu}^{(3)} \equiv f_{\mu,\nu} - A_{\mu\nu}^{(1)} - A_{\mu\nu}^{(2)}$. A flow is purely of type 3, i.e., $f_{\mu,\nu} = A_{\mu\nu}^{(3)}$, if and only if the proper density is uniformly constant through space and time.

Proof.—$f \equiv$ const implies $f_{\mu,\nu}f_\nu + f_\mu f_{\nu,\nu} = 0$ by (3), and contraction of the latter relation with f_μ yields $f_{\nu,\nu} = 0$, which makes the quantities (12) and (13) vanish.

To obtain a flow-vector solution by integrating the differential form associated with (6),

$$df_\mu = \sum_{i=1}^{3} A_{\mu\nu}^{(i)}\, dx_\nu, \qquad (14)$$

one must fix the quantities ξ, η_μ, and $\zeta_{\mu\nu}$ in (7)–(9) in a manner that satisfies the exterior product integrability conditions[8]

$$d(df_\mu) = \sum_{i=1}^{3} dA_{\mu\nu}^{(i)} \wedge dx_\nu = 0 \qquad (15)$$

that follow from (14). The analytical procedure for satisfying (15) is often straightforward, as illustrated by the example of purely type-1 flows, i.e., those for which $\eta_\mu \equiv 0 \equiv \zeta_{\mu\nu}$. For such flows (14) and (7) produce

$$df_\mu = \xi\, dx_\mu + (2-\alpha)^{-1}(5-\alpha)\xi f^{-2}f_\mu(f\cdot dx) \qquad (16)$$

and by contraction of (16) with f_μ one obtains

$$f df = 3(2-\alpha)^{-1}\xi(f\cdot dx). \qquad (17)$$

The latter relation can be used to eliminate $(f\cdot dx)$ from (16), which becomes

$$df_\mu = \xi\, dx_\mu + \tfrac{1}{3}(5-\alpha)f_\mu f^{-1}\, df. \qquad (18)$$

Hence (15) states that

$$d\xi \wedge dx_\mu + \tfrac{1}{3}(5-\alpha)f^{-1}\, df_\mu \wedge df = 0 \qquad (19)$$

because $d\alpha \wedge df = (d\alpha/df)df \wedge df = 0$. By substituting (18) into the second exterior product on the left side of (19), one gets

$$[d\xi - \tfrac{1}{3}(5-\alpha)\xi f^{-1}\, df] \wedge dx_\mu = 0, \qquad (20)$$

which implies that

$$\xi = \xi_0 f^{5/3}\exp[-\tfrac{1}{3}\int_0^f \alpha(\lambda)\lambda^{-1}d\lambda] \qquad (21)$$

with $\xi_0 \equiv$ const. The general integral to (18) is therefore

$$f_\mu = \xi(x_\mu - k_\mu), \qquad (22)$$

where k_μ denotes a constant vector of integration. Finally, one obtains an implicit equation for f as a function of the space-time coordinates by squaring (22):

$$(x-k)\cdot(x-k) = -f^2\xi^{-2} = -\xi_0^{-2}f^{-4/3}\exp[\tfrac{2}{3}\int_0^f \alpha(\lambda)\lambda^{-1}d\lambda]. \qquad (23)$$

This general solution for purely type-1 flows depicts a radially symmetric disturbance in the fluid that either expands from or implodes to the singularity at $x_\mu = k_\mu$. Directly expressible in terms of the primary variables by the recalling of (2), (4), and (5), the exact solution (22) and (23) can be specialized for relativistic astrophysical phenomena of contemporary interest.[7] In the classical limit $c \to \infty$, the solution (22) and (23) satisfies the spherical wave equations of classical compressible flow theory.[9]

For a broad class of more general flows the task of satisfying (15) is facilitated by the following.

Theorem.—For *symmetric flows* characterized by $\zeta_{\mu\nu} \equiv \zeta_{\nu\mu}$, the flow vector is expressible as

$$f_\mu = f\{\exp[-\int_0^f \alpha(\lambda)\lambda^{-1}d\lambda]\}\phi_{,\mu} \qquad (24)$$

with ϕ a real scalar function of the space-time coordinates.

Proof.—The substitution of (7)–(9) into (14) produces

$$df_\mu = \xi\, dx_\mu + [\alpha\eta_\mu + (2-\alpha)^{-1}(5-\alpha)\xi f^{-2}f_\mu]w_f + f_\mu w_\eta + \zeta_{\mu\nu}dx_\nu, \qquad (25)$$

where

$$w_f \equiv (f\cdot dx), \quad w_\eta \equiv (\eta\cdot dx) \qquad (26)$$

are differential forms. By taking the contracted exterior product of (25) and dx_μ, one obtains[10]

$$df_\mu \wedge dx_\mu = (1-\alpha)\, w_\eta \wedge w_f \tag{27}$$

for $\zeta_{\mu\nu} \equiv \zeta_{\nu\mu}$. Thus

$$dw_f = df_\mu \wedge dx_\mu = [(1-\alpha)\, w_\eta] \wedge w_f, \tag{28}$$

which by the Cartan-Frobenius integration theorem[8] implies that

$$w_f = \psi\, d\phi \tag{29}$$

for certain real scalar functions ψ and ϕ. The former function is determined by contracting (25) with f_μ to get

$$w_\eta = f^{-1}\, df - 3(2-\alpha)^{-1}\xi f^{-2} w_f \tag{30}$$

and substituting (29) and (30) into (28); the resulting equation yields

$$\psi = f\{\exp[-\int_0^f \alpha(\lambda)\lambda^{-1}\, d\lambda]\} \tag{31}$$

to within a multiplicative factor that can be absorbed into the definition of ϕ. Hence (24) follows from (29) and (31).

Corollaries.—The quantity η_μ is given by (30) and (24) as

$$\eta_\mu = f^{-1} f_{,\mu} - 3(2-\alpha)^{-1}\xi f^{-1}\{\exp[-\int_0^f \alpha(\lambda)\lambda^{-1}\, d\lambda]\}\phi_{,\mu}. \tag{32}$$

But since $\eta_\nu f_\nu \equiv 0$ according to (8), (32) implies that

$$\xi = -\tfrac{1}{3}(2-\alpha)\{\exp[-\int_0^f \alpha(\lambda)\lambda^{-1}\, d\lambda]\}\phi_{,\nu} f_{,\nu}, \tag{33}$$

where use has been made of the square of (24);

$$\phi_{,\nu}\phi_{,\nu} = -\exp[2\int_0^f \alpha(\lambda)\lambda^{-1}\, d\lambda]. \tag{34}$$

It is easy to verify that (32) and (33) are consistent with (13) and (12).

Expression (24) is a general intermediate integral to (25) for symmetric flows. In the following representative symmetric flow solutions, ϕ and f have been determined by *Ansatz* to satisfy (34) and the integrability conditions (15).

(a) Purely type-2 flows ($\xi \equiv 0 \equiv \zeta_{\mu\nu}$) with

$$\phi = R_0 \tanh^{-1}[(a\cdot x + k)/(b\cdot x + k')], \tag{35}$$

and f given implicitly by

$$R_0 \exp[-\int_0^f \alpha(\lambda)\lambda^{-1}\, d\lambda] = R \equiv [(b\cdot x + k')^2 - (a\cdot x + k)^2]^{1/2} \tag{36}$$

in which a_μ and b_μ are constant orthogonal timelike and spacelike unit vectors ($a\cdot a = -1$, $a\cdot b = 0$, $b\cdot b = 1$) and R_0, k, k' are scalar constants. By putting (35) into (24), one finds

$$f_\mu = [a_\mu \cosh(\phi/R_0) - b_\mu \sinh(\phi/R_0)]f \tag{37}$$

and (25) is satisfied exactly with $\xi \equiv 0 \equiv \zeta_{\mu\nu}$ and

$$\eta_\mu = \alpha^{-1} R^{-1}\left[a_\mu \sinh\left(\frac{\phi}{R_0}\right) - b_\mu \cosh\left(\frac{\phi}{R_0}\right)\right]. \tag{38}$$

The remarkable feature of these purely type-2 symmetric flows is that they are admissible for arbitrary $p = p(\rho)$ relations, as manifest in the disposability of the function $\alpha(f)$.

(b) Type 1+2 flow ($\zeta_{\mu\nu} \equiv 0$) with

$$\phi = \tfrac{1}{2}k\tau \tan^{-1}[k(a\cdot x)/(x\cdot x)], \tag{39}$$

f given implicitly by

$$2(x\cdot x)\left[\left(\frac{a\cdot x}{x\cdot x}\right)^2 + k^{-2}\right] = \tau \exp[-\int_0^f \alpha(\lambda)\lambda^{-1}\, d\lambda], \tag{40}$$

and $\alpha(f)$ such that

$$(2-\alpha)^{-1}(1-2\alpha)\exp[-\int_0^f \alpha(\lambda)\lambda^{-1}\, d\lambda] = \tau^{-1} \equiv \text{const.} \tag{41}$$

In (39) and (40) a_μ is a constant timelike unit vector ($\vec{a}\cdot\vec{a} = -1$) and k is a scalar constant. The substitution of (39) into (24) produces

$$f_\mu = [a_\mu - 2(x\cdot x)^{-1}(a\cdot x)x_\mu]f \tag{42}$$

and (25) is satisfied exactly with $\zeta_{\mu\nu} \equiv 0$,

$$\xi = -2f(x\cdot x)^{-1}(a\cdot x), \tag{43}$$

$$\eta_\mu = 2\alpha^{-1}(x\cdot x)^{-1}\{(x\cdot a)a_\mu - [1+2(x\cdot x)^{-1}(a\cdot x)^2]x_\mu\}. \tag{44}$$

In order for (41) to hold for all f, the equation of state must be such that

$$\alpha^{-3}(2-\alpha)^{-1}(1-2\alpha)^4 = \text{const}\times f^2, \tag{45}$$

as found by differentiating the logarithm of (41) and integrating the resulting equation. Thus, the simple flow (42) of type $1+2$ requires a rather complicated equation of state.

Equations (25) and (15) can also be solved by *Ansatz* for nonsymmetric flows characterized by $\zeta_{\mu\nu} \neq \zeta_{\nu\mu}$. For example, in the case of constant-density purely type-3 flows ($\xi \equiv 0 \equiv \eta_\mu$), (25) reduces to

$$df_\mu = \zeta_{\mu\nu}\, dx_\nu, \tag{46}$$

and exact solutions are readily obtainable for $\zeta_{\mu\nu}$ that satisfy the conditions in (9). Two examples are the following.

(a) Constant-density vortex flows with

$$\begin{aligned} f_\mu &= k_\mu + l_\mu \cos\theta + m_\mu \sin\theta, \\ \zeta_{\mu\nu} &= (m_\mu \cos\theta - l_\mu \sin\theta)\theta_{,\nu}, \end{aligned} \tag{47}$$

in which k_μ, l_μ, m_μ are mutually orthogonal constant vectors subject to the conditions $k\cdot k < 0, l\cdot l = m\cdot m > 0$, $k\cdot l = l\cdot m = m\cdot k = 0$, and θ is a scalar space-time function which varies in the fourth space-time direction: $k_\nu \theta_{,\nu} = l_\nu \theta_{,\nu} = m_\nu \theta_{,\nu} = 0$.

(b) Constant-density shear flows with

$$f_\mu = k_\mu \cosh\omega + l_\mu \sinh\omega, \tag{48}$$

$$\zeta_{\mu\nu} = (k_\mu \sinh\omega + l_\mu \cosh\omega)\omega_{,\nu}, \tag{49}$$

in which k_μ, l_μ are mutally orthogonal constant vectors subject to the conditions $-k\cdot k = l\cdot l > 0$, $k\cdot l = 0$, and ω is a scalar space-time function which may vary in the two space-time directions orthogonal to k_μ and l_μ: $k_\nu \omega_{,\nu} = l_\nu \omega_{,\nu} = 0$. With the quantity (5) identically constant, these analytical solutions for purely type-3 flows are the relativistic correspondents of the constant-density and constant-pressure vortex and shear flows of classical compressible theory, for which $\vec{u} = \text{const} \times (\cos\theta(x_3), \sin\theta(x_3), 0)$ and $\vec{u} = \text{const}\times(0, 0, \sinh\omega(x_1, x_2))$ in particular Galilean frames of reference.

These representative exact solutions illustrate the efficiency of the analytical solutional method. Since this Cartan-Frobenius integration method is principally rooted in the quasilinearity of Eqs. (13) and can be generalized for extra inhomogeneous terms (that do not contain a highest-order partial derivative), it would appear that similar treatments are applicable to the higher-dimensional forms of many quasilinear NPDE of practical importance.

[1]S. Kumei and G. W. Bluman, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **42**, 1157 (1982).

[2]G. Rosen, Phys. Rev. Lett. **49**, 1844 (1982).

[3]The classical hodograph transformation is discussed in a revealing general context in Ref. 1.

[4]Here Greek indices run 1, 2, 3, 4 with the summation convention for repeated indices understood. The fourth components of vectors are purely imaginary, and commas followed by subscripts denote differentiation with respect to the space-time coordinates.

[5]Since the classical nonrelativistic fluid flow equations are obtained from (1) in the formal limit $c \to \infty$, the results in this Letter can be transcribed immediately for the nonrelativistic limiting case. Four-dimensional symmetry makes the more general relativistic equations (1) particularly amenable to the integration method reported here.

[6]The region of applicability of a relation $p = p(\rho)$ has been delineated by the thermodynamic analyses of M. A. Herrera and S. Hacyan, Phys. Fluids **26**, 1446 (1983); J. Gabriel, Nuovo Cimento **74B**, 167 (1983); M. A. Cirit, Lett. Nuovo Cimento **30**, 350 (1981).

[7]M. H. Johnson and C. F. McKee, Phys. Rev. D **3**, 858 (1971); P. G. Eltgroth, Phys. Fluids **14**, 2631 (1971); C. L. Pekeris, Proc. Natl. Acad. Sci. U. S. A. **73**, 687 (1976), and Proc. Roy. Soc. London, Ser. A **355**, 53 (1977); P. J. Wiita, Astrophys. Space Sci. **54**, 407 (1978); I. Lerche and P. J. Wiita, Astrophys. Space Sci. **68**, 207, 475 (1980).

[8]É. Cartan, *Systémes Differérentiels Extérieurs et Leurs Applications Géométriques,* Actualités Scientifiques Industrielles, No. 994 (Hermann, Paris, 1945); H. Flanders, *Differential Forms* (Academic, New York, 1963), pp. 92–102. The calculus of differential forms has been employed heretofore in various treatments of differential equations, e.g., B. K. Harrisand and F. B. Estabrook, J. Math. Phys. **12**, 653 (1971); C. P. Boyer and E. G. Kalnins, J. Math. Phys. **18**, 1032 (1977); W. Strampp, W. H. Steeb, and W. Erig, Prog. Theor. Phys. **68**, 731 (1982), and works cited therein.

[9]For example, R. von Mises, *Mathematical Theory of Compressible Fluid Flow* (Academic, New York, 1958), p. 86.

[10]Here use is made of the antisymmetry property of the exterior product in the relations $\eta_\mu w_f \wedge dx_\mu = w_f \wedge w_\eta \equiv -w_\eta \wedge w_f$, $f_\mu w_f \wedge dx_\mu = w_f \wedge w_f = 0$, and $\zeta_{\mu\nu} dx_\mu \wedge dx_\nu \equiv \frac{1}{2}(\zeta_{\mu\nu} - \zeta_{\nu\mu})\, dx_\mu \wedge dx_\nu$.