# Theory of Ponderomotive Stabilization of a Magnetically Confined Plasma

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A self-consistent description is obtained for the low-frequency evolution of a plasma, the electromagnetic field that confines it, and the amplitude of a high-frequency antenna-driven field. The ponderomotive energy of the system is related to the antenna reactive energy. Equilibrium equations and a variational principle for stability are given in terms of ion and electron ponderomotive forces, magnetization current, and self-consistent modification of the high-frequency field.

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Recent experiments on the Phaedrus tandem mirror at Wisconsin[1] have been performed with two rf antennas emitting in the ion gyrofrequency range: The first antenna is used for ion cyclotron resonance heating, and the second, which emits somewhat above the ion gyrofrequency, is used for stabilization. The quadrupole fields are turned off, so that the mirror field is axisymmetric. It is observed that the level of fluctuation is low, which seems to indicate good confinement properties, and that the power deposited by the second antenna is negligible. When, on the other hand, the voltage of the second antenna is reduced to zero, the fluctuations in the plasma increase dramatically, reflecting the flute instability. This experiment suggests that the stabilization by the field of the second antenna is a ponderomotive effect, unrelated to dissipation.

The explanation that is customarily given[2–6] to explain the stabilizing effect of the high-frequency field is based on the consideration of ion drifts. It is argued that when the ion ponderomotive drift balances the curvature and magnetic-gradient drifts, then the plasma is stabilized. Furthermore, since the expression for the ion ponderomotive force exhibits a resonance at the ion gyrofrequency, one deduces that these forces get very large close to that frequency.

We show in this Letter that a self-consistent treatment leads to very different conclusions. Indeed, the high-frequency-field characteristics such as polarization and propagation depend on the plasma configuration, and act in turn on the plasma through ponderomotive forces and magnetization current. It will be seen that there is no actual singularity at the ion gyrofrequency, and that the equilibrium and self-consistent stability analysis must include *all* of the effects: electron and ion ponderomotive forces, magnetization current, and perturbation of the high-frequency field pattern.

A convenient way to take all the reciprocal interactions into account is to consider a global picture of the system which includes the plasma, the electromagnetic field, and the antenna. A complete description is provided by the Lagrangian action of the system, expressed in terms of the displacement of the individual plasma particles and in terms of the electromagnetic vector potential (we adopt the radiation gauge for convenience). Variation of the action with respect to the displacements yields, of course, the Newton-Lorentz equations, while variation with respect to the vector potential gives the Maxwell equations including the antenna and plasma current sources.

Since the frequency $\omega$ of the field emitted by the antenna is of the order of the ion gyrofrequency and is much larger than the rate $\gamma$ at which the flute instability develops, a separation of time scales is appropriate. We represent the motion $\vec{r}_{tot}$ of the particles as the sum of a low-frequency motion (the motion $\vec{r}_{oc}$ of the oscillation center) and of a high-frequency oscillation with amplitude $\vec{r}$ modulated at low frequency:

$$\vec{r}_{tot}(t)=\vec{r}_{oc}(t)+\mathrm{Re}[\vec{r}(t)\exp(-i\omega t)].$$

We use a similar representation for the field: The vector potential $\vec{A}_{tot}$ is the sum of a low-frequency component $\vec{A}_{lf}$ and of a high-frequency component of amplitude $\vec{A}$:

$$\vec{A}_{tot}(\vec{x},t)=\vec{A}_{lf}(\vec{x},t)+\mathrm{Re}[\vec{A}(\vec{x},t)\exp(-i\omega t)].$$

The action is then expressed in terms of these variables, expanded up to second order in the amplitudes, and averaged over the fast time scale, under the assumption that no resonance takes place. The expansion is based on the assumption of small interchange forces, which result from the unfavorable *average* curvature, or in the model developed later, from the gravitational potential $\Psi$. Formally, we use the ordering $\vec{A}=O(\eta)$, $\gamma=O(\eta)$, $\Psi=O(\eta^2)$, with $\eta$ the small parameter. The new form of the action is now the sum of three distinct contributions: the oscillation-center action, the average field action, and the ponderomotive action that collects all the terms second order in the high-frequency amplitudes.

The total ponderomotive energy $V$, that is associated with the ponderomotive action so derived, turns out to have the simple expression[7]

$$V=-\frac{1}{16\pi}\frac{\omega^2}{c^2}\int d^3x\left\{\vec{A}^*\cdot\overleftrightarrow{\epsilon}\cdot\vec{A}-\frac{c^2}{\omega^2}|\nabla\times\vec{A}|^2\right\}-\frac{1}{2c}\int d^3x\,\mathrm{Re}(\vec{j}_a^*\cdot\vec{A}), \tag{1}$$

where $\overleftrightarrow{\epsilon}(\vec{x})$ is the Hermitian (since we exclude resonances) local dielectric tensor at $\omega$, and $\vec{j}_a$ is the antenna current density amplitude. The equation satisfied by the high-frequency field $\vec{A}$ is obtained by setting the variation of the expression (1) equal to zero for all variations of $\vec{A}$, and is the driven wave equation

$$(\omega^2/c^2)\overleftrightarrow{\epsilon}\cdot\vec{A}-\nabla\times(\nabla\times\vec{A})=-(4\pi/c)\vec{j}_a.$$

The solution $\vec{A}$ and the ponderomotive energy $V$ are to be considered as functionals of the plasma densities and magnetic fields, which appear in the dielectric tensor $\overleftrightarrow{\epsilon}$.

It is easy to show that the value of $V$, at its stationary point, is half the last term of Eq. (1). It is therefore equal to the reactive energy of the antenna:

$$V=L|I|^2/4, \tag{2}$$

where $L$ is the inductance of the antenna in the presence of the plasma, and $I$ is the given amplitude of the antenna current. Since the value of the inductance is measurable experimentally, the above interpretation of $V$ allows the experimental determination of the (generalized) ponderomotive force $F$ on the plasma, if one relates the variations of the inductance $\Delta L$ to the (generalized) plasma displacement $\Delta a$, and uses the equation $F=-(|I|^2/4)\Delta L/\Delta a$. The relation (2) will also be very useful for optimal antenna design, as we shall see when we examine the stability conditions.

In the cold-plasma limit, the dielectric tensor assumes its usual expression[8] and is a local function of the oscillation-center densities $n_s$ ($s$ is species label) and of the low-frequency magnetic field $\vec{B}$. The variation of the averaged action with respect to its variables leads to a closed set of two-fluid equations for oscillation-center densities and momentum densities. The ponderomotive effects appear, first, as ponderomotive-force densities

$$-n_s\nabla(\delta V/\delta n_s)$$

in the momentum equations, and second, as a magnetization-current density

$$-c\nabla\times(\delta V/\delta\vec{B})$$

in Ampère's law. More explicitly, the ponderomotive potential and the magnetization are given by functional derivatives of $V$[7,9,10]:

$$\frac{\delta V}{\delta n_s}(\vec{x})=-\frac{1}{16\pi}\frac{\omega^2}{c^2}\vec{A}^*\cdot\frac{\partial\overleftrightarrow{\epsilon}}{\partial n_s}\cdot\vec{A},$$

and

$$-\frac{\delta V}{\delta\vec{B}}(\vec{x})=\frac{1}{16\pi}\frac{\omega^2}{c^2}\vec{A}^*\cdot\frac{\partial\overleftrightarrow{\epsilon}}{\partial\vec{B}}\cdot\vec{A}.$$

To be definite, we shall neglect here the component of $\vec{A}$ parallel to $\vec{B}$ (a consequence of small electron mass), and use suitable approximations of the components $S$ and $D$ of the dielectric tensor[10]:

$$S=-\omega_{pi}^2/(\omega^2-\Omega_i^2),$$

and

$$D=\omega_{pi}^2\Omega_i/\omega(\omega^2-\Omega_i^2)-\omega_{pe}^2/\Omega_e\omega,$$

where $\omega_{ps}$ and $\Omega_s$ are the plasma frequencies and the signed gyrofrequencies. As a result, we get

$$\frac{\delta V}{\delta n_s}(\vec{x})=-\frac{1}{16\pi}\frac{\omega^2}{c^2}\left\{\frac{\partial S}{\partial n_s}|\vec{A}|^2-i\frac{\partial D}{\partial n_s}\vec{A}^*\times\vec{A}\cdot\hat{b}\right\},$$

where $\hat{b}$ is the unit vector in the $\vec{B}$ field direction, as well as a similar equation for the magnetization.

Although the derivatives of the dielectric tensor in the above equations do indeed exhibit poles at the ion gyrofrequency, the ponderomotive potential and magnetization current are *not singular* at that frequency.[11,12] This is a result of self-consistency that imposes a special polarization to the high-frequency field: $\vec{A}$ is the solution of the wave equation, and, close to the ion gyrofrequency, is circularly polarized in the electron direction. As will become clear below [Eq. (3)], a singularity does appear (for $0<|k_\parallel|<\infty$), but not at $\Omega$; rather it is on the Alfvén-ion-cyclotron branch of the dispersion relation:

$$\omega=k_\perp v_A[1+(k_\parallel v_A/\Omega_i)^2]^{1/2},$$

where $v_A$ is the Alfvén speed. The properties of the waves and the existence of the resonance make it essential to consider the $k_{\parallel}$ spectrum emitted by the antenna. Since the resonance is always *below* the ion gyrofrequency, its sudden absence above $\Omega_i$ is consistent with the sharp transition that is observed experimentally. According to the expression (3), the change in stability across $\Omega_i$ is not due to a sudden reversal of all the ponderomotive forces. It might rather be associated with the local energy deposition produced by the resonance.

Comparison of the electron and ion ponderomotive potentials shows that the former often dominates the latter, and in any case cannot be neglected in the analysis. Note that for a quasineutral variation

$$\delta n_e = \delta n_i = \delta n,$$

the variation of $V$ is the sum of an electron and an ion contribution, i.e.,

$$\delta V = (\delta V/\delta n_e + \delta V/\delta n_i)\delta n.$$

The equations simplify in the two-dimensional magnetohydrodynamic (MHD) approximation, with the magnetic field in the $\hat{z}$ direction. The oscillation-center density $n(\vec{x})$ and the magnetic field $B(\vec{x})$ are then purely convected by the velocity field $\vec{u}(\vec{x})$:

$$\partial n/\partial t + \nabla\cdot(n\vec{u}) = 0,$$

and

$$\partial B/\partial t + \nabla\cdot(B\vec{u}) = 0.$$

The momentum equation is

$$nM(\partial\vec{u}/\partial t + \vec{u}\cdot\nabla\vec{u}) = -\nabla(B^2/8\pi + p_\perp) - n\nabla\Psi - n\nabla(\delta V/\delta n) - B\nabla(\delta V/\delta B),$$

where $M$ is ion mass, $p_\perp$ is perpendicular pressure,[13] and $-\nabla\Psi$ is an outward gravitational field which mimics unfavorable average magnetic field curvature. Note the ponderomotive and magnetization contributions in the equation. Finally, $\vec{A}$ obeys

$$(\omega^2/c^2)(S\vec{A} - iD\vec{A}\times\hat{z}) - \nabla\times(\nabla\times\vec{A}) = -(4\pi/c)\vec{j}_a,$$

which completes the system of equations.

When the plasma and the antenna are both axisymmetric, the static equilibrium equation expresses radial pressure balance:

$$\frac{d}{dr}\left(\frac{B^2}{8\pi} + p_\perp\right) + n\frac{d}{dr}\left(\frac{\delta V}{\delta n} + \Psi\right) + B\frac{d}{dr}\left(\frac{\delta V}{\delta B}\right) = 0.$$

For that geometry, one can eliminate the radial component of $\vec{A}$, to obtain

$$\frac{\delta V}{\delta n}(\vec{x}) = -\frac{1}{16\pi}\frac{\omega^2}{c^2}|A_\theta|^2\frac{\partial}{\partial n}\frac{[\omega_{pi}^2 + N_{\parallel}^2\,\Omega_i(\omega-\Omega_i)][\omega_{pi}^2 - N_{\parallel}^2\,\Omega_i(\omega+\Omega_i)]}{\Omega_i^2[\omega_{pi}^2 + N_{\parallel}^2(\omega^2-\Omega_i^2)]}, \qquad (3)$$

and a similar expression for the magnetization. Here

$$N_{\parallel}^2 = k_{\parallel}^2 c^2/\omega^2.$$

A simple evaluation shows that magnetization and ponderomotive force have comparable contributions and must both be included.

From the linearization of the equations around their equilibrium, one can build a modified delta-$W$ variational principle that allows an easy practical determination of the stability. For instance, for an incompressible displacement field (presumably the most unstable flute motion)

$$\vec{\xi} = \nabla\zeta\times\hat{z},$$

with stream function

$$\zeta(r,\theta,t) = \hat{\zeta}(r)\exp(i,m,\theta)\exp(\gamma t),$$

one obtains[7] the modified delta-$W$ variational principle:

$$\gamma^2 = -(W_1 + W_2 + W_3)/N. \qquad (4)$$

Here

$$N = \int d^3x\, Mn\left[\left|\frac{d}{dr}\hat{\zeta}\right|^2 + \frac{m^2}{r^2}|\hat{\zeta}|^2\right]$$

is a measure of the plasma intertia,

$$W_1 = -\int d^3x\left[\frac{m^2}{r^2}|\hat{\zeta}|^2\frac{dB}{dr}\frac{d}{dr}\left(\frac{\delta V}{\delta B}\right)\right]$$

is the magnetization energy contribution, and

$$W_2 = -\int d^3x\left[\frac{m^2}{r^2}|\hat{\zeta}|^2\frac{dn}{dr}\frac{d}{dr}\left(\frac{\delta V}{\delta n} + \Psi\right)\right]$$

is the interchange contribution modified by the equilibrium ponderomotive force. The last term of

Eq. (4),

$$W_3=\int\int d^3x\, d^3x'\, \frac{m^2}{rr'}\hat{\zeta}^*(r)\hat{\zeta}(r')\left[\frac{\delta^2 V}{\delta n\,\delta n'}\frac{dn}{dr}\frac{dn'}{dr'}+2\frac{\delta^2 V}{\delta n\,\delta B'}\frac{dn}{dr}\frac{dB'}{dr'}+\frac{\delta^2 V}{\delta B\,\delta B'}\frac{dB}{dr}\frac{dB'}{dr'}\right],$$

comes from the distortion of the high-frequency field due to the flute motion of the plasma.[14] The second-order functional derivatives of $V$ are nonlocal quantitites (i.e., depend on two points $r$ and $r'$), and their expressions involve the high-frequency plasma-wave Green's function for the equilibrium configuration.[7]

The relation of $V$ with the antenna reactive energy [Eq. (2)] allows the following interpretation: If the plasma configuration is such that the antenna impedance is minimum, then the distortion effects are stabilizing. In fact, for the simple case of axisymmetric high-frequency field evanescent in the plasma, it turns out that distortion effects are destabilizing. A field rotating in the angular direction might be more favorable because it tends to be naturally (i.e., in vacuum) more localized outside of the plasma region.

Preliminary estimates show that the required rf energy that is necessary to stabilize the plasma is of the order of the free energy that is liberated by the plasma during interchange of the flux tubes. More precise estimates need to take into account delicate geometric features of the mirror and of the antenna, and will be the object of future research.

In summary, we have derived a complete set of equations for plasma dynamics with ponderomotive effects, and we have explicit expressions for the ponderomotive forces and magnetization current. Self-consistency is essential and deeply affects the conclusions. In particular (in the cold-plasma model) there is no singularity at the ion gyrofrequency, and the consideration of waves with finite $k_\parallel$ is important. The equilibrium and stability analysis must include ion and electron ponderomotive forces, magnetization current, and distortion of the high-frequency field. We gave the modified delta-$W$ variational principle in a simple MHD case, and analyzed the effect of distortion on stability. We showed the relation of the ponderomotive energy to the antenna inductance, and described the possible applications to experimental measurements of ponderomotive effects and to optimal antenna design. Details of the derivations and generalization of the results given here will appear in a forthcoming publication.[7]

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