Dense Matter in the Chiral Soliton Model

M. Kutschera
Institute of Nuclear Physics, 31-342 Kraków, Poland, and NORDITA, DK-2100 Copenhagen Ø, Denmark

and

C. J. Pethick
NORDITA, DK-2100 Copenhagen Ø, Denmark, and Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801

and

D. G. Ravenhall
Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801
(Received 25 May 1984)

We explore some properties of dense matter in Skyrme's chiral soliton model and show that at high densities the energy density varies as $n^{4/3}$, where n is the baryon density. This is quite different from the behavior of conventional nuclear models, but is very similar to that of quark matter. The maximum mass of neutron stars constructed of such matter is significantly lower than that for most other versions of dense matter and may account for the absence of neutron stars in many extended supernova remnants. The phase transition to quark matter is expected to be softened considerably by such a nuclear-matter model.

PACS numbers: 14.80.Dq, 11.10.Lm, 97.60.+z

In a series of prescient papers Skyrme[1,2] showed how to construct baryonlike objects as topological excitations of a meson field. In recent years, identification of the topological density with the baryon density has been placed on a firmer footing by the work of Witten.[3] The model has been successful in accounting for properties of single baryons[2,4] and in describing the gross features of the interactions among nucleons.[2,5,6] To date, most discussion of that problem has been devoted to the two-body interaction. In this paper we explore the predictions of the Skyrme model for a many-body problem, namely dense matter.

The basic Skyrme Lagrangian density is

$$\tilde{L} = -\frac{f_\pi^2}{4}\,\mathrm{Tr}[L_\mu L_\mu] - \frac{\epsilon^2}{4}\,\mathrm{Tr}[L_\mu, L_\lambda]^2, \qquad (1)$$

where $L_\mu = U^\dagger \partial_\mu U$, f_π ($\simeq 93$ MeV) is the pion decay constant, and ϵ is a constant determining the strength of the fourth-order term introduced to stabilize the chiral soliton. The unitary matrix U, the basic ingredient of the theory, specified by a direction $\hat{n}$ and a chiral rotation angle θ, is related to the singlet meson field σ and the triplet (pion) field $\vec{\pi}$ by

$$U = e^{i\vec{\tau}\cdot\hat{n}\theta} = f_\pi^{-1}[\sigma(\vec{x}) + i\vec{\tau}\cdot\vec{\pi}(\vec{x})], \qquad (2)$$

where $\vec{\tau}$ are the usual Pauli isospin matrices.

Let us first investigate the energy. As in previous calculations of nuclear forces we consider static configurations and neglect quantum fluctuations. We consider a U of the form $U(\vec{r}/\lambda)$, where λ is some scale length. The spatial components of the current L therefore scale as λ^{-1}. At high densities the dominant term in the Lagrangian density is the fourth-order one, which scales as λ^{-4}, whereas the second-order term scales as λ^{-2} and may therefore be neglected at high enough densities. Thus for small λ, the energy density scales as λ^{-4}. On the other hand, the average baryon number density n scales as λ^{-3}. Thus we conclude that at high densities (small λ) the energy density E must vary as $n^{4/3}$. The result is implicit in the work of Skyrme,[2] who obtained a lower bound on the energy proportional to $n^{4/3}$.

Now let us make quantitative estimates of the energy of skyrmion matter. At low densities matter will consist of a number of essentially noninteracting skyrmions. At the center of each skyrmion U takes on the value -1. The simplest skyrmion has a simple hedgehoglike structure, with the pion field in the radial direction, $\hat{n} = \hat{r}$ in Eq. (2). Other skyrmions may be obtained from this one by subjecting it to a (space-independent) isospin rotation. As the density increases, skyrmions begin to overlap, but there are still points at which $U = -1$, which may be regarded as the positions of the skyrmions. The energy of a configuration depends on the positions of the skyrmions and the isospin rotation characteristics of the structure near these points. One can of course envisage structures in which one has *lines* or *surfaces* on which $U = -1$, but to consider them here would take us too far afield, and we shall confine our attention to configurations which resemble approximately spherical skyrmions, modified by their mutual overlap.

Our problem therefore is to solve for the chiral field for a given configuration of skyrmions. This is a horrendous variational problem, so we shall make some approximations. First we consider only the case when all skyrmions are identical. Second we replace the many-skyrmion problem by a single-skyrmion problem with a boundary condition, in the spirit of the Wigner-Seitz approximation in solid-state physics. The skyrmion is assumed to be at the center of a spherical cell whose volume is equal to the average volume per skyrmion, and we impose the condition that the chiral angle vanish everywhere on the surface of the sphere. To motivate this approximation, consider an array of identical hedgehog skyrmions, all in the defensive position, on a simple cubic lattice, as sketched in Fig. 1. It is easy to see that, by symmetry, the chiral angle vanishes on the corners of the unit cell, at the centers of the edges, and at the face centers, a total of 26 points. Since the Lagrangian depends on gradients of the fields, the fact that $\theta = 0$ at so many points on the surface of the unit cell means that θ cannot vary from zero greatly over the whole surface. Consequently it should be a reasonable approximation to put $\theta = 0$ at all points on the surface. (We note that, by the variational principle, imposing this constraint will give an energy greater than the actual one.) Similar arguments apply to other lattices and to disordered configurations of skyrmions, provided that in them no skyrmions are much closer together than the typical skyrmion spacing. The assumption that the environment of a skyrmion is spherical should be reasonable, since each skyrmion is surrounded by a rather large number of neighbors.

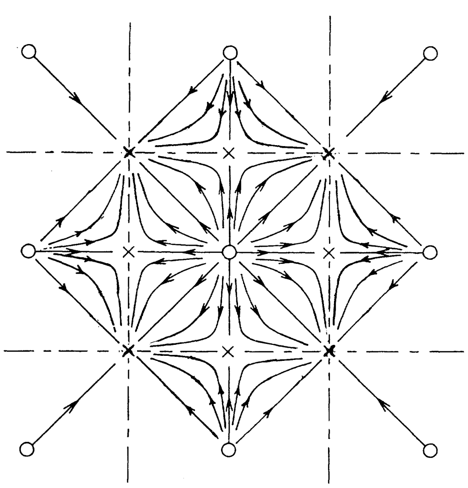

FIG. 1. Isospin directions of the pion field in a unit cell of a cubic lattice of skyrmions, on a plane containing the centers. Skyrmion centers ($U = -1$) are indicated by small circles, and points corresponding to the vacuum state ($U = 1$) by crosses.

The chiral angle is given by the same equations as for an isolated skyrmion,[5,6] except that the boundary condition that θ tend to zero at large distances in the free-skyrmion case is replaced by the condition $\theta = 0$ at a radius r equal to the cell radius r_c. For $r \to 0$, θ tends to π for a skyrmion with baryon number unity, just as in the free skyrmion case. At high densities (small r_c) we find that $E/\hbar c \approx 870\epsilon^2 n^{4/3}$. This is to be compared with the lower bound $E/\hbar c > 12(2\pi^2)^{4/3}\epsilon^2 n^{4/3} = 640\epsilon^2 n^{4/3}$ obtained by Skyrme.[2] The rather small difference between our estimate and Skyrme's lower bound is remarkable, since our calculation is for the most repulsive configuration possible, where all skyrmions are identical. This shows that the energy can be reduced by no more than 25% by the isospin "combing" of hedgehogs. Physically this means that at high density isospin-dependent correlations have little effect on the energy.

In Fig. 2(a) we plot the result for the interaction energy per unit volume of skyrmion matter, that is, the total energy density minus the baryon density times the free skyrmion mass energy. In our numerical calculation we took for ϵ^2 the value 0.005 52, as obtained by Jackson and Rho[5] by fitting to the experimental value of g_A. This gives an isolated skyrmion mass of 1420 MeV. For comparison we show results for more conventional descriptions of matter, those of Bethe and Johnson[7] using the Reid nucleon-nucleon potential and of Friedman and Pandharipande[8] using the V_{14} potential plus a three-nucleon interaction. The energy of the skyrmions is greater than those of the traditional models at low densities because we have chosen the most repulsive skyrmion configuration and have calculated the energy variationally. At high density the energy of the skyrmions, which is proportional to $n^{4/3}$, increases less rapidly than that of the more traditional models.

The $n^{4/3}$ dependence of the energy per unit volume is radically different from that of any conventional model of nuclear matter in which two-nucleon interactions dominate.[7,8] In the latter case the energy density must vary as n^2, as in Ref. 7, or some higher power if spin-orbit forces are included, as in Ref. 8. Since the arguments on which these conventional nuclear models are based (nonrelativistic dynamics, local potentials, hard or soft

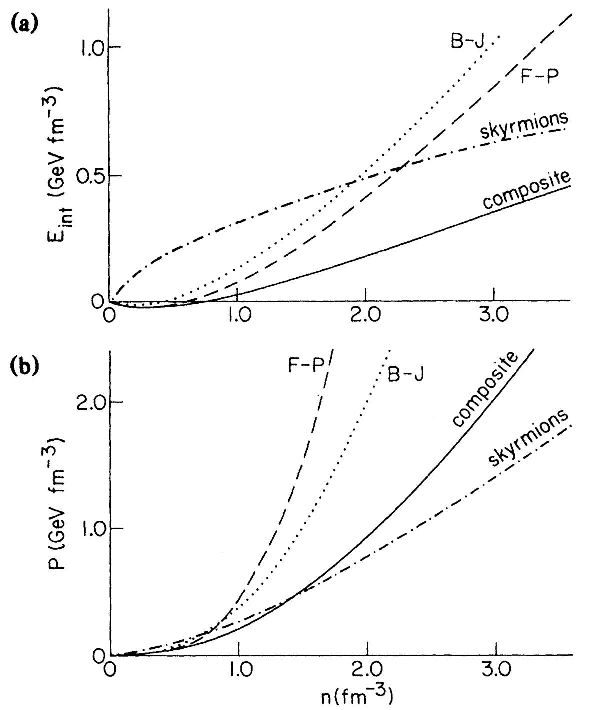

FIG. 2. (a) The energy per unit volume as a function of baryon density, for pure skyrmions, for the calculations of Bethe and Johnson (Ref. 7) and of Friedman and Pandharipande (Ref. 8), and for our composite model (see text). (b) Pressure as a function of baryon density, for the same cases as (a).

cores) lose their validity at high densities, those n dependences are in no way compelling. The Skyrme model is therefore a valuable one for providing clues as to the nature of the nucleon-nucleon interaction at high densities. As Skyrme pointed out,[2] the $n^{4/3}$ dependence of the energy density implies an $n^{1/3}$ or $1/r_c$ dependence of the energy per skyrmion. This may be understood as being due to a $1/r$ potential acting between neighboring skyrmions. The reduction of the interaction between other than nearest neighbors comes about because of the essential nonlinearity of the short-distance part of the Skyrme Lagrangian (1). This $1/r_c$ dependence of the interaction energy is *not* in disagreement with the result[5] that for large N the spherically symmetric N-skyrmion state has interaction energy proportional to N^2. This is because that N-skyrmion state is an onionlike arrangement of baryons, each having the form of a spherical shell, quite unlike the roughly spherical baryons of the dense matter we are considering. This will be discussed more fully elsewhere.[9]

We now investigate consequences for neutral-star properties. First we must allow for the kinetic energy of the skyrmions. We shall assume that this is given by the kinetic energy of an ideal neutron gas. The result for the total pressure as a function of baryon density is shown in Fig. 2(b), where for comparison we also show the results of Refs. 7 and 8. We have mocked up an equation of state which interpolates between the Friedman and Pandharipande results[8] near nuclear density (where we believe they are the most accurate representation of neutron matter) and the skyrmion results at high density. Extrapolation of the results of Ref. 8 was made with an analytic fit to the tabulated quantities so that the energy per unit volume increased as n^2 at high density.[10] To obtain a composite expression with the desired skyrmion high-density behavior this has been reduced by a factor $1+(n/n_0)^{2/3}$; the value $n_0=2.185$ fm^{-3} gives the requisite coefficient of $n^{4/3}$.

The maximum neutron-star mass with this composite equation of state is $1.51M_0$, where M_0 is the solar mass. The radius is 8.7 km and the central density is 1.8 fm^{-3}. In the vicinity of the maximum, the mass is rather insensitive to the central density, and the neutron-star mass of $1.40M_0$ occurs at a radius of 9.8 km and a central density of 1.1 fm^{-3}. This maximum mass is substantially lower than those obtained from most equations of state. (The corresponding mass for Ref. 8 is $1.96M_0$.) This reduction depends to some extent on the manner in which the equations of state of Ref. 8 and of skyrmions have been combined in the composite model, but it is clear that the trend must be in this direction. If in a stellar collapse the condensed remnant has a mass greater than the maximum neutron-star mass it will presumably form a black hole. Since many stellar collapse calculations lead to condensed remnants of mass about $1.4M_0$, it is quite likely that many of the remnants could be black holes. If this were the case, it would provide an explanation for neutron stars being found in only a few supernova remnants.[11] The maximum mass we find is not inconsistent with observational estimates of neutron-star masses.[12] However, we stress the fact that there are still many problems in this area.

It is interesting to compare the energy of skyrmion matter to that of quark matter. We assume three flavors of massless quarks. At high densities the limiting behavior of E_q is given to first order in the QCD coupling constant α by

$$E_q = B + \tfrac{9}{4}\pi^{2/3}\left(1+\frac{2\alpha}{3\pi}\right)n^{4/3}\hbar c$$

$$\sim 4.38\left(1+\frac{2\alpha}{3\pi}\right)n^{4/3}\hbar c, \qquad (3)$$

where B is the bag constant. The coupling constant α is probably considerably less than the MIT value[13] 2.2, and therefore Eq. (4) very likely lies between 4.83 and 7.08 times $n^{4/3}$. This is to be compared with the limiting behavior of the skyrmions, which including their Fermi energy is $E_S \simeq (2.32+870 \times \epsilon^2) n^{4/3}\hbar c$, which is $7.15 n^{4/3}\hbar c$ for $\epsilon^2 = 0.005\,52$. For $\epsilon^2 = 0.004\,21$, the value used by Adkins, Nappi, and Witten,[4] one finds $E_S = 5.98 n^{4/3}\hbar c$. We therefore come to the remarkable conclusion that at high densities the energy of the skyrmion matter is equal to that of asymptotically free quarks to within the uncertainties in the calculations. We strongly suspect that one of the reasons for the phenomenological success of the Skyrme model in accounting for the properties of baryons is that the fourth-order term mocks up the properties of a bag of quarks. Further support for this view comes from recent calculations in the chiral bag model,[14] made up of quarks contained by the outer part ($\theta \lesssim \pi/2$) of a skyrmion. For a bag of radius 0.44 fm, nucleon properties can be explained quite satisfactorily without invoking the fourth-order term in the skyrmion part of the model.

In view of the fact that skyrmion matter at high densities seems to be a good approximation to a quark gas (and is probably modeling the quark gas), it does not make much sense to discuss phase transitions between skyrmions and quarks. A physical way to discuss the transition to quark matter would be to extend our calculations to the chiral bag model. We suspect that the transition between bags of quarks with chiral field in between and a phase where quarks fill all of space would be accompanied by a smaller density change than the corresponding transition involving a conventional nuclear model. We therefore expect that in relativistic heavy-ion collisions it will be difficult to detect the transition to quark matter through its influence on the equation of state.

We wish to thank Hans Bethe, Gerry Brown, Wieslaw Czyż, Andy Jackson, Igor Mishustin, and Mannque Rho for many useful discussions. We also thank our colleagues at the University of Illinois, especially Vijay Pandharipande, for their help and interest. This work was supported in part by the M. Skłodowska-Curie Foundation through Grant No. F7-071-P, and by the National Science Foundation under Grants No. NSF PHY82-01948 and NSF PHY80-25605.

[1]T. H. R. Skyrme, Proc. Roy. Soc. London **247**, 260 (1958), and **252**, 236 (1959).

[2]T. H. R. Skyrme, Proc. Roy. Soc. London **260**, 127 (1961), and **262**, 237 (1961), and Nucl. Phys. **31**, 556 (1962).

[3]E. Witten, Nucl. Phys. **B223**, 422 (1983).

[4]G. S. Adkins, C. R. Nappi, and E. Witten, Nucl. Phys. **B228**, 552 (1983).

[5]A. D. Jackson and M. Rho, Phys. Rev. Lett. **51**, 751 (1983).

[6]A. Jackson, A. D. Jackson, and V. Pasquier, to be published.

[7]H. A. Bethe and M. Johnson, Nucl. Phys. **A230**, 1 (1974), as tabulated by W. D. Arnett and R. L. Bowers, *Report of Nuclear and Astrophysical Group* (University of Texas, Austin, 1974).

[8]B. Friedman and V. R. Pandharipande, Nucl. Phys. **A361**, 502 (1981).

[9]C. J. Pethick and M. Kutschera, to be published.

[10]D. G. Ravenhall, unpublished; D. G. Ravenhall, C. J. Pethick, and C. Lorenz, to be published.

[11]D. J. Helfand and R. H. Becker, Nature (London) **307**, 215 (1984).

[12]P. C. Joss and S. A Rappaport, in *Accretion Driven Stellar X-Ray Sources,* edited by W. H. G. Lewin and E. P. J. van den Heuvel (Cambridge Univ. Press, Cambridge, 1984), p. 1.

[13]T. DeGrand, R. L. Jaffe, K. Johnson, and T. Kiskis, Phys. Rev. D **12**, 2060 (1975).

[14]G. E. Brown, A. D. Jackson, M. Rho, and V. Vento, Phys. Lett. **140B**, 285 (1984).