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Solvable Integrodifferential Equations and Their Relation to the Painlevé Conjecture

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The Ablowitz-Ramani-Segur conjecture relates the Painlevé property, i.e., polelike local singularity structure, to integrability. As such this conjecture cannot be applied, as integrability detector, to nonlocal integrodifferential equations. We show, however, that, for the physically interesting cases of the Benjamin-Ono and intermediate long-wave equations, the Painlevé property can be recovered despite the apparent nonlocality.

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The discovery of integrable partial differential equations (PDE) has spurred the development of a whole new branch of nonlinear physics. The pioneering work of Kruskal and Zabusky[1] on the Korteweg–de Vries (KdV) equation was soon followed by the proof that this equation possessed an infinite number of conservation laws and that it could be integrated using inverse scattering techniques (IST).[2] A host of integrable nonlinear PDE's have been discovered to date. Although the integration through IST has not yet been performed for all of them the integrability of a given equation appears guaranteed as soon as a Lax pair[3] can be obtained for it. However, even the derivation of a Lax-pair representation for a given PDE can sometimes be a very hard task and thus the necessity for an integrability criterion arose. A conjecture, due to Ablowitz, Ramani, and Segur,[4] has related integrability to the Painlevé property. They have conjectured, and amply verified, that whenever a PDE is integrable by IST all its reductions to ordinary differential equations (ODE) have only poles as movable (i.e., initial-condition-dependent) singularities of their solutions.

There exist, however, equations to which the Painlevé criterion does not apply, one might say, "by definition": This is the case of the integrodifferential equations where, because of nonlocality of the integral kernel, a local singularity analysis does not make sense in principle. Two such equations are well known describing the propagation of long internal waves in a stratified fluid: the Benjamin-Ono (BO) equation:

$$u_t + uu_x + \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{u_{xx}(x')}{x-x'} dx' = 0, \tag{1}$$

and the intermediate-long-wave (ILW) or finite-depth equation:

$$u_t + uu_x - \tfrac{1}{2} P \int_{-\infty}^{+\infty} u_{xx}(x') \coth \tfrac{1}{2}\pi (x-x')\, dx' = 0. \tag{2}$$

Both are integrable. For the BO equation some conservation laws and solitary-wave solutions were given by Benjamin[5] and Ono[6] themselves. Chen, Lee, and Pereira[7] have presented a N-soliton solution, while Kruskal and Bock[8] have given the Lax-pair representation. The inverse-scattering transform for this equation has only recently been given by Fokas and Ablowitz.[9] It is of particular interest as it presents common features with the IST for multidimensional PDE's. The ILW equation has been proposed by Joseph[10] based on an equation due to Whitham[11] together with a dispersion relation associated to a "thin thermocline."[12] He presented solitary and multisoliton[13] solutions. The same equation has been derived on a somewhat different approach by Kubota, Ko, and Dobbs[14] and Segur and Hammack.[15] Chen and Lee have given the N-soliton solution[16] to the ILW equation. Kodama, Ablowitz, and Satsuma[17] have presented the associated scattering problem, and in fact performed the inverse-scattering transform. Recently Degasperis and Santini[18] have presented the construction of whole hierarchies of integrable equations, the lowest-order member of which are the BO and the ILW equations.

The integrability of these two equations does not "*a priori*" seem related to any kind of Painlevé criterion because of nonlocality. The main result of this paper is to show how, guided by the underlying physical problem, one can rewrite the BO and ILW equations so as to make the singularity analysis applicable to them.

The two equations have a common origin. They describe long internal wave propagation in the presence of weak dispersion and weak nonlinearities. However, while the ILW equation describes propagation in a fluid of finite total thickness D, the BO equation makes the assumption of infinite thickness ($D \to \infty$). Incidentally, the shallow-fluid limit ($D \to 0$) of ILW is just the KdV equation. It is clear from this description that the problem of the wave propagation for $D \neq 0$ is essentially a $(2+1)$-dimensional one. In fact, going back to the derivation of the equations, one obtains immediately that the velocity potential must satisfy the Laplace equation in the layer of thickness D. Calling it u ($v_x = \partial u/\partial x$, $v_y = \partial u/\partial y$) we have

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0, \tag{3}$$

with boundary conditions $u = f(x,t)$ at $y=0$ and $\partial u/\partial y = 0$ at $y = -D$ for ILW or $u=0$ at $u = -\infty$ for BO. Written in $2+1$ dimensions the BO and ILW equations assume the common form

$$u_t + uu_x + u_{xy} = 0 \quad (\text{at } y=0). \tag{4}$$

So both equations are described by the system (3) and (4) with only boundary conditions distinguishing between them. We can show briefly how to recover the usual form for the BO equation. It suffices to show that

$$u_y = \frac{1}{\pi}\frac{\partial}{\partial x} P\int_{-\infty}^{+\infty} \frac{f(x',t)}{x'-x}dx' \quad \text{at} \quad y=0, \tag{5}$$

when u is a solution of (3). In fact the solution of the Laplace equation subject to the boundary condition $u = f(x,t)$ at $y=0$ and $u=0$ at $y=-\infty$ is readily obtained as

$$u(x,y,t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x',t)\frac{y}{y^2+(x-x')^2}dx'. \tag{6}$$

This implies that

$$u_y(x,y,t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x',t)\frac{(x-x')^2-y^2}{[y^2+(x-x')^2]^2}dx' = \frac{1}{\pi}\frac{\partial}{\partial x}\int_{-\infty}^{+\infty} f(x',t)\frac{x'-x}{y^2+(x-x')^2}dx'. \tag{7}$$

Taking the limit at $y \to 0$ introduces a principal value of the integral so

$$u_y(x,y,t)\,|_{y=0} = \frac{1}{\pi}\frac{\partial}{\partial x}P\int_{-\infty}^{+\infty}\frac{f(x',t)}{x'-x}dx'. \tag{8}$$

The proof for the ILW equation proceeds along exactly the same lines starting from an expression different from (6), due to the different boundary conditions $\partial u/\partial y = 0$ at $y = -D$.

One important point must be stressed here. The procedure we introduced above is quite general. It is not restricted to the BO and the ILW equations (in which case it would be of little interest as both equations are known to be integrable). It applies in fact to any integrodifferential equation involving the Hilbert integral transform or the more general one present in Eq. (2). For instance, all the equations of the Degasperis-Santini hierarchy can be treated along the same lines. The critical point is that whenever u is a harmonic function [i.e., a solution of Eq. (3)], its y derivative at $y=0$ satisfies

$$u_y = Tu_x,$$

where T is the Hilbert or the coth transform depending on the boundary conditions for u. This allows the transcription of any integrodifferential equation involving the T transform to a system of partial differential equations with one extra dimension. (In the case where the T transform applies on the function u itself and not its x derivative, it suffices to introduce w such that $u=w_x$ and apply our formalism to w.)

Before proceeding with the singularity analysis let us show how the shallow depth limit ($D\rightarrow 0$), i.e., the KdV equation, can be recovered as a singular limit of system (3) and (4). Let us consider a very small depth $D=\epsilon$. In that case we can expand

$$\left.\frac{\partial u}{\partial y}\right|_{y=-D}=0=\left.\frac{\partial u}{\partial y}\right|_{y=0}-\epsilon\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0}+\frac{1}{2}\epsilon^2\left.\frac{\partial^3 u}{\partial y^3}\right|_{y=0}+O(\epsilon^3).$$

From (3) we have $\partial^2 u/\partial x^2=-\partial^2 u/\partial y^2$. Substituting in (4) we obtain

$$u_t+uu_x-\epsilon\frac{\partial^3 u}{\partial x^3}+\frac{\epsilon^2}{2}\frac{\partial^4 u}{\partial x\,\partial y^3}+O(\epsilon^3)=0. \quad (9)$$

By the appropriate scaling $t\rightarrow t/-\epsilon$, $u\rightarrow -u\epsilon$ the equation becomes

$$u_t+uu_x+u_{xxx}+O(\epsilon)=0, \quad (10)$$

and the limit $\epsilon\rightarrow 0$ reduces it to KdV.

We will now perform the singularity analysis for BO and ILW written in the form (3),(4). We remark first that the general solution of Eq. (3) is $u=f(x+iy,t)+g(x-iy,t)$. This means that from (3) any singularity is allowed provided that it propagates on the characteristics $x+iy=\phi(t)$ and $x-iy=\chi(t)$. We investigate now the constraints on the singularities due to (4). For this, we follow a method which does away with reductions to ODE's and works on the PDE itself.[19,20] Let us consider the singularity manifold $\psi=x+iy-\phi(t)$ where $\phi(t)$ is a free function of t. If we look for leading-order singularities of the form $u=a_0/\psi^\alpha$ we readily find $a_0=2i$, $\alpha=1$. We now expand u around this singularity:

$$u=g(x-iy,t)+\frac{2i}{\psi}(1+a_1\psi+a_2\psi^2+\dots), \quad (11)$$

where $a_i=a_i(t)$. (It is straightforward to convince oneself that the "resonances" of this equation are -1 and 2, which means that no terms further than a_2 will be needed in this expansion.) We compute the derivatives u_x, u_{xy} and then expand g around the singularity as $g=g|_{\psi=0}+\psi g'|_{\psi=0}+\dots$ [$g'=\partial g(s,t)/\partial s$]. The limit $y\rightarrow 0$ is then taken, which gives

$$g=g(x=\phi(t),t)+\phi g'(x=\phi(t),t)+\dots.$$

If we balance the various powers of ψ from expansion (11) in Eq. (4), we obtain

$$2ia_1+g=\dot{\phi}, \quad (12)$$

where $g=g(\phi(t),t)$, which defines a_1, and $2i\times(2ia_2+g')-2i(2ia_2+g')=0$, i.e., the consistency relation at the "resonance" for a_2 is automatically verified. This means that a_2 is a free function of t and the expansion (11) is indeed of Painlevé type. This completes the singularity analysis for Eqs. (3) and (4).

At this point two remarks are in order. First the equation (4) by itself is *not* of Painlevé type (even at $y=0$). In fact when Eq. (4) alone is considered, the singularity manifold around which we expand is of the form $\psi=x+\omega(y,t)$ with $\omega(y,t)$ an otherwise unspecified function of y and t. In that case we obtain, for a leading singularity of the type $u=a_0/\psi^\alpha$, $a_0=2\omega_y$, and $\alpha=1$. If we expand u in the form

$$u=2\omega_y/\psi+a_1(y,t)+\psi a_2(y,t)+\dots, \quad (13)$$

we successively obtain

$$\omega_t+\omega_{yy}/\omega_y+a_1=0, \quad (14)$$

which defines a_1 and $\omega_{yt}=0$ as a compatibility condition at the next order. Clearly this is inconsistent with the initial assumption that ω is a free function of y and t. So Eq. (4) alone is not of Painlevé type. [Note that the condition $\omega_{yt}=0$ is automatically satisfied when (3) is obeyed as in that case $\omega(y,t)=\pm iy+\phi(t)$].

The second remark concerns the application of this procedure to the equations which are not intrinsically integrodifferential, as for example the KdV equation. In fact the KdV can also be written in 1+2 dimensions in the form

$$\begin{aligned}&u_{xx}+u_{yy}=0,\\&u_t+uu_x-u_{xyy}=0 \quad (\text{at } y=0).\end{aligned} \quad (15)$$

So, for a singularity manifold of the form $\psi=x+iy-\phi(t)=0$, the expansion we are seeking reads

$$u=\frac{2}{\psi^2}[1+\sum_{n\geq 1}a_n(t)\psi^n]+2g(x-iy,t).$$

As in the BO-ILW case we first compute u_t, u_x, and u_{xyy} and then expand g in powers of ψ:

$$g=\sum_{n\geq 0}\{g^{(n)}(x=\phi(t),t)/n!\}\psi^n.$$

Let us now incorporate the $g^{(n)}$ in the a_n coefficients through

$$b_n(t) = a_n(t) + \{g^{(n-2)}(\phi(t),t)/(n-2)!\}.$$

The analysis then goes through as in the usual KdV form for the b_n coefficients. This is true because, as the number of the y derivatives is even, the relative sign of a_n and $g^{(n-2)}$ is the same whether arising from the $(u_t + uu_x)$ term or from the u_{xyy} term. This is not the case for the BO-ILW equation. There the number of y derivatives is odd, the relative sign of a_n and $g^{(n-2)}$ is different whether arising from $(u_t + uu_x)$ or u_{xy}, and these two quantities cannot be merged in a single coefficient b_n. This just reflects the fact that in (15) the nonlocality is entirely fictitious while in the case of (3) and (4), it is essential in order to eliminate the y variable.

Finally let us note that the analysis does not distinguish between BO and ILW. Indeed the analysis is purely local, while the only difference between these two equations comes from boundary conditions which are transparent to a Painlevé analysis.

As a conclusion we can remark that the underlying physics has been a most useful guide to the problem we have examined. Although the reduced versions of the BO and ILW equations were not tractable by the standard singularity analysis methods, as soon as their physical derivation was examined the extension to higher dimensionality became evident and this allowed an application of the Painlevé criterion. Moreover the transcription of the singular integral equations to a system of a linear plus a nonlinear equation has led to a most original application of the Painlevé analysis. Thus the present work confirms, once again, the great value of the singularity analysis as a tool for the investigation of the integrability of dynamical systems.

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