Universal Localized Relaxation Mode of Uniformly Driven Kinks in Damped Multistable Systems

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Linear arrays of damped multistable systems in a constant driving field F are considered in the continuum limit. The existence of a universal localized relaxation mode ("inertia mode") of the driven kinks is explicitly proven. This mode, of frequency $\omega = -i\eta/m$, collapses in the undamped ($\eta \to 0$) free ($F \to 0$) chain into the Goldstone mode of the corresponding "free kink," and in a chain without inertia ($m \to 0$) it relaxes instantaneously.

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Linear arrays of uniformly[1-10] or periodically[11-15] driven and linearly damped multistable systems (mainly modeled by the sine-Gordon or ϕ^4 potential) have attracted in recent years increased attention. As a result of the interplay and competition between nonlinearity, damping, and driving force, fascinating effects can occur.[1-15] My aim in the present paper is to describe a new interesting phenomenon of this kind, which consists of the occurrence of a universal, localized, smooth relaxation mode of the uniformly driven kinks (domain walls) in linearly damped and tightly coupled multistable lattice-dynamical and similar systems. As mentioned in the Abstract, the existence of this universal relaxation mode is intrinsically connected to the inertia of the damped multistable chain. On this ground, it will be referred to hereafter in this paper as the "inertia mode" of the driven kinks.

Let us consider therefore a linear chain of particles subjected to a constant external force F and described by the classical Hamiltonian

$$H=\sum_i\{\tfrac{1}{2}m\dot{\phi}_i{}^2+\tfrac{1}{2}k(\phi_{i+1}-\phi_i)^2+V(\phi_i)-F\phi_i\}.$$

Here m is the particle mass, $\phi_i(t)$ represents the displacement of the ith particle at the time t from the site x_i of a reference lattice, $\dot{\phi}_i=\partial\phi_i/\partial t$, k is the strength of the harmonic coupling between neighboring particles, and $V(\phi_i)$ is a general "multistable" on-site potential. We assume that the chain deformation changes gradually from one lattice site to the next (tightly coupled particles), so that the continuum theory applies. We also assume that each particle is subjected to a damping force proportional to its velocity. In this way we obtain the following equation of motion for the displacement field $\phi(x,t)$:

$$m\ddot{\phi}+\eta\dot{\phi}-kl^2\phi''=F-dV/d\phi. \tag{1}$$

Here η is the damping coefficient, l is the lattice constant, the prime denotes $\partial/\partial x$, and all the quantities are expressed in physical units.

In the absence of the driving field and dissipation, Eq. (1) describes a multistable Hamiltonian system. The stable steady states $\phi_n{}^0$ correspond to degenerate minima[16] of the potential $V(\phi)$, i.e., they are degenerate classical vacuums[17] of the displacement field. These uniform configurations are equally favored and are separated by transition regions called (static) kinks, solitons, or domain walls. Because of the "relativistic" invariance of the corresponding equation of motion (1), the kinks separating two adjacent domains can also uniformly move along the chain, with any velocity smaller than the velocity of sound $c=(kl^2/m)^{1/2}$. If a constant external field F is switched on, then, for limited intensities,[4] $F<F_{\max}$, the system still remains multistable, but the domains become unequally favored and the domain walls accelerate to acoustic velocities, while their width goes to zero. If, however, at the same time damping is present, the domain walls may survive as permanent profile excitations of the displacement field, moving with a unique terminal velocity[1-7] $v<c$. The inertia mode of these driven kinks represents my main concern in this paper.

The driven kinks under consideration are traveling solitary waves $\phi_K=\phi(X)$, $X=x-vt$, which interpolate the displacement field for $X\to\pm\infty$ between the stable uniform configurations $\phi_{1,2}$ corresponding to any two adjacent relative minima of the "bias potential" $V(\phi)-F\phi$. In other words, the kink driven with a uniform velocity $|v|<c$ satisfies the equation

$$W_0\phi_K''+\eta v\phi_K'-V'(\phi_K)+F=0 \tag{2}$$

and the boundary conditions

$$\phi_K(\pm\infty)=\phi_{1,2},\quad \phi_K'(\pm\infty)=0, \tag{3}$$

where $W_0=kl^2-mv^2>0$ and the prime denotes derivative with respect to the argument.

In order to examine the linear excitations of the driven kink we first transform Eq. (1) to the comoving frame of the kink by $(x, t) \to (X, t)$, such that $\partial/\partial x \to \partial/\partial X$, $\partial/\partial t \to \partial/\partial t - v\partial/\partial X$, and then linearize the transformed equation around ϕ_K according to the *Ansatz*

$$\phi(x, t) = \phi_K(X) + \varphi(X)\exp(-i\omega t), \tag{4}$$

where φ is an infinitesimal deviation from the kink. The obtained equation, transcribed in standard Sturm-Liouville form, looks like

$$(e^{\alpha X}\varphi')' + W_0^{-1}\{m\omega^2 + i\eta\omega - V''(\phi_K)\}e^{\alpha X}\varphi = 0, \tag{5}$$

where $V''(\phi_K) = d^2V(\phi)/d\phi^2$ for $\phi = \phi_K$ and

$$\alpha = \eta v W_0^{-1}(1 - 2im\eta^{-1}\omega). \tag{6}$$

I am interested in localized modes of the kink, defined as eigensolutions of Eq. (5) satisfying the boundary conditions $\varphi(\pm\infty) = 0$ and normalizable in the Sturm-Liouville sense, i.e.,

$$N = \int_{-\infty}^{+\infty} e^{\alpha X}|\varphi(X)|^2 dX = \text{finite}. \tag{7}$$

The wave equation (2) is invariant with respect to the translation of the X coordinate. In the kink $\phi_K(X)$, however, this symmetry is broken. Therefore, the zero-frequency translation mode[16-18] (Goldstone mode)

$$\omega_T = 0, \quad \varphi_T(X) = \phi_K'(X), \tag{8}$$

which restores the broken translation symmetry, must be an eigenmode of the driven kink. Indeed, it is easy to check that the corresponding equation (5),

$$\varphi_T'' + \alpha_0\varphi_T' - W_0^{-1}V''(\phi_K)\varphi_T = 0, \tag{9}$$

where $\alpha_0 = \alpha(\omega = 0) = \eta v W_0^{-1}$, is identically satisfied. It is, however, not immediately obvious that the translation mode is also localized, i.e.,

$$N_T = \int_{-\infty}^{+\infty} \exp(\alpha_0 X)\phi_K'^2(X)dX = \text{finite}. \tag{10}$$

There are two essential things which ensure the finiteness of N_T: (1) the boundary conditions (3) which imply $\varphi_T(\pm\infty) = 0$, and (2) the positive curvature $V''(\phi_{1,2})$ of the potential $V(\phi)$ in the asymptotic states $\phi_{1,2}$ (which also implies the stability of $\phi_{1,2}$). Indeed, if we consider, e.g., the kink moving in the positive X direction (i.e., $v > 0$), then for $X \to +\infty$, φ_T^2 decays exponentially with a decay length $\{\alpha_0 + [\alpha_0^2 + 4W_0^{-1}V''(\phi_1)]^{1/2}\}^{-1}$, i.e., the integrand in (10) decays for $X \to +\infty$ as $\exp\{-[\alpha_0^2 + 4W_0^{-1}V''(\phi_1)]^{1/2}X\}$. Therefore, because of the mentioned two conditions the translation mode (8) is really localized. [If ϕ_1 were an unstable steady state, corresponding to a relative maximum of the bias potential, i.e., $V''(\phi_1) < 0$, the normalizability of φ_T would be questionable.] On the other hand, $\phi_K(X)$ is a monotonic function of X, and thus $\varphi_T(X)$ is nodeless. Hence $\omega_T = 0$ represents the lowest eigenvalue of (5) which implies the linear stability of the driven kinks against localized, as well as against extended, perturbations of the dynamics.

I can now formulate and prove succinctly the point of this Letter, namely that

$$\omega_I = -i\eta m^{-1}, \quad \varphi_I(X) = \exp(\alpha_0 X)\varphi_T(X) \tag{11}$$

represents a universal, localized, smooth relaxation mode, the inertia mode of the driven kink ϕ_K. Indeed by substituting (11) in (5) and (6), one obtains for φ_T just Eq. (9), i.e., (ω_I, φ_I) is really an eigensolution. Having in mind the asymptotic behavior of φ_T, one immediately sees that the boundary conditions $\varphi_I(\pm\infty) = 0$ are also satisfied. The inertia mode is universal in the sense that ω_I is model (i.e., potential) independent and the functional structure of $\varphi_I(X)$ is also independent of the on-site potential V. When we take into account that $\alpha(\omega = \omega_I) = -\alpha_0$, it results immediately that the norm (7) of the inertia mode (11) equals the norm (10) of the translation mode, $N_I = N_T$, which shows that in the comoving frame of the kink, φ_I is really a localized mode. Furthermore, for $t \to +\infty$, the "neighboring" solution

$$\phi_K + \varphi_I\exp(-i\omega_I t) = \phi_K + \varphi_I\exp(-\eta m^{-1}t)$$

relaxes smoothly to the kink as claimed above. Thus, the smaller the mass (i.e., the inertia) of the particles is, the faster is the relaxation. At the limit $m \to 0$ (chain without inertia) the relaxation to the kink becomes instaneous, i.e., the inertia mode disappears.

Equation (11) shows that the spatial part of the inertia mode is obtained by a monotonic exponential modulation of the translation mode. Let us now discuss the rate coefficient $\alpha_0 = \eta v(kl^2 - mv^2)^{-1}$ of this modulation. One sees that α_0 depends essentially on the kink velocity. Having in mind the boundary conditions (3), this "nonlinear eigenvalue" v of Eq. (2) may be expressed by the integral formula

$$v = \{V(\phi_1) - V(\phi_2) - F(\phi_1 - \phi_2)\} \times \left(\eta\int_{-\infty}^{+\infty}\phi_K'^2 dX\right)^{-1}. \tag{12}$$

This is an implicit equation for v and thus it is difficult to see how v really depends on the driving field F. For $F \to 0$, however, as argued by Landauer[19] in the case of the (bistable) ballast resistor, to first order in F, for any multistable system

$$\lim_{F\to 0}\{V(\phi_1)-V(\phi_2)-F(\phi_1-\phi_2)\}=\Lambda F$$

holds, where Λ is a constant. In this way, to first order in F, the kink velocity may be calculated[19] by substituting into (12) instead of ϕ_K the wave function $\phi_K{}^0(x)$ of the static kink interpolating between the degenerate vacua $\phi_{1,2}{}^0=\lim_{F\to 0}\phi_{1,2}$ of the free chain, i.e.,

$$v=\Lambda F[\eta\int_{-\infty}^{+\infty}(\phi_K{}^{0\prime})^2dx]^{-1}. \quad (13)$$

Therefore, if $F\to 0$, the Landauer formula (13) implies that the rate coefficient α_0 in (11) is proportional to F, i.e., α_0 vanishes for $F=0$. Thus, in the limit $\eta\to 0$, $F\to 0$ (undamped, free chain), the inertia mode (11) collapses into the Goldstone mode (8) of the corresponding free kink.

The situation with which we are faced with here is the famous "defect case" of a degenerate eigenvalue. For $F=0$ and $\eta=0$, $\omega=0$ is a doubly degenerate eigenvalue, but we have only a single eigenfunction with exponential t dependence. This is the Goldstone mode $\varphi_T(x)\exp(0t)$ of the static free kink. The "defect-causing" solution is the nonexponential ("algebraic") "mode" $t\varphi_T(x)$ which describes a kink moving with infinitesimal velocity. If damping is switched on ($\eta\neq 0$, $F=0$) the degeneracy of $\omega=0$ is lifted by the occurrence of the inertia mode, which describes the relaxation of infinitesimally slowly moving kinks to the initial static configuration.

A more transparent picture of the v-F characteristic of the uniformly driven kinks may be obtained with the aid of the Büttiker-Thomas formula,[6,9] which is equivalent to (12), and looks like

$$v=\pm v_0 l(\chi^2+Q^{-2})^{-1/2}. \quad (14)$$

Here $v_0=(kV_0)^{1/2}/\eta$, $\chi=(mV_0)^{1/2}/\eta$, V_0 represents a conveniently defined "strength" of the potential $V(\phi)$, and Q (denoted in Refs. 6 and 9 by ϕ) is in fact the single model-specific quantity which determines the kink velocity via the potential V as a function of the ratio $r=F/V_0$. In this way we get

$$\alpha_0=\pm(V_0/kl^2)^{1/2}Q(1+\chi^2Q^2)^{1/2}. \quad (15)$$

Therefore, the rate coefficient α_0 depends in a universal fashion on the Büttiker-Thomas (model-specific) function $Q(r)$. In the weak-field limit $F\to 0$, one has[6,9] $Q\propto r$ and thus, one recovers the above result $\alpha_0\propto F$.

I would like to end this Letter with some explicit results concerning the inertia mode of the driven kinks in multistable systems modeled by the popular sine-Gordon potential $V=V_0(1-\cos\phi)$. For more concreteness I shall consider as physical background a Josephson-active transmission line[20] and an analog mechanical system.[10]

In the case of the sine-Gordon potential, for $F<F_{\max}$, the driven kink solution to Eq. (2) is unfortunately not available in closed analytic form. In the weak-field limit, however, the terminal velocity may be obtained immediately from the Landauer formula (13), which yields $v=(\pi/4\eta)(kl^2/V_0)^{1/2}F$, and thus the rate coefficient α_0 in the inertia mode (11), to first order in F, becomes $\alpha_0=(\pi/4)(kl^2V_0)^{-1/2}F$. The corresponding frequency ω_I is given by the universal expression (11).

A Josephson transmission line which includes a distributed bias source and in which the only dissipation is due to the conduction current of normal electrons across the barrier may be described[10,20] by Eq. (2) with the mentioned sine-Gordon potential. In this case ϕ represents the quantum mechanical phase difference across the junction and the other quantities in (2) have to be replaced as follows[10]: $m\to\phi_0C/2\pi$, $kl^2\to\phi_0/2\pi L$, $\eta\to g\phi_0/2\pi$, $F\to j_B$, and $V_0\to J_c$, where $\phi_0=h/2e$ is the flux quantum, L and C are the series inductance and the shunt capacitance per unit length of the line, respectively, J_c is the maximum current per unit length that the junction line will pass, and g and j_B are the shunt conductance and the distributed bias source per unit length of the line, respectively. On the ground of the above results we can, therefore, predict that the localized perturbations of the driven kinks transmitted along a Josephson-active line decay exponentially in time with a rate $|\omega_I|=g/C$. Here the shunt resistance per unit length g^{-1} is a measure of the damping and the shunt capacitance per unit length measures the "inertia" of the Josephson line. The rate coefficient α_0 in the inertia mode is now given by

$$\alpha_0=(\pi/4)(\phi_0/2\pi LJ_c)^{-1/2}(j_B/J_c).$$

In the mechanical analog of the Josephson-active line constructed by Nakajima, Sawada, and Onodera,[10] the decay rate $|\omega_I|$ corresponds to the ratio $\tilde{k}/I$, where $\tilde{k}$ is the damping constant of a single disk and I is the moment of inertia of a single pendulum plus a disk. By using the numerical results reported in the first paper of Ref. 10, I obtained relaxation times $|\omega_I|^{-1}$ of the perturbed mechanical kinks between 2.63 and 58 sec. As in these experiments the kink velocity increases from zero to its limiting value 21.5 cm/sec and the rate coefficient α_0 takes values between 0 and 0.02 π cm^{-1}.

Finally, I would like to emphasize that the existence of the inertia mode is also an imperative requirement of physical intuition.

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