Improved Theory of the Muon Anomalous Magnetic Moment

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A new theoretical value $a_\mu = 11\,659\,202(20)\times 10^{-10}$ is reported for the muon anomalous moment, based on the first complete calculation of the α^4 QED term and improvements of the α^3 QED term and various hadronic contributions. The remaining error is mostly due to the experimental inputs needed for evaluation of the hadronic vacuum polarization effect. Further improvement of this error seems possible. Thus the electroweak theory may be tested at the one-loop level if measurement of a_μ is improved by an order of magnitude.

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The anomalous magnetic moment a_μ is one of the basic properties of the muon which is measurable with great precision and also calculable from theory. Thus it provides a sensitive tool for testing the validity of the theoretical framework. In early days it served as a testing ground of QED. More recently it has been used for detection of the hadronic vacuum polarization effect. It has also been used to obtain information about the possible internal structure of the muon,[1] along with useful constraints on supersymmetric theory.[2] We wish to demonstrate in the following that its role as an important probe of the electroweak effect is just around the corner.

The most accurate measurements thus far of a_μ are those obtained in 1977 at the CERN muon storage ring[3]:

$$\begin{aligned} a_{\mu^-} &= 11\,659\,370(120)\times 10^{-10}, \\ a_{\mu^+} &= 11\,659\,110(110)\times 10^{-10}, \end{aligned} \tag{1}$$

where the numerals enclosed in parentheses represent the uncertainties in the final digits of the measured values. The best theoretical estimate reported prior to this article is[4]

$$a_\mu^{\text{th}} = 11\,659\,213(100)\times 10^{-10} \tag{2}$$

in good agreement with (1).

In contrast to the electron anomaly a_e which is dominated by the QED effect, a_μ is much more sensitive to physics at smaller distances because of its larger mass scale. Thus a_μ^{th} of (2) has a substantial contribution ($\sim 7\times 10^{-8}$) from the hadronic vacuum polarization effect. Even the effect of the weak interaction is not negligible. Using the latest information[5] on the Weinberg angle and the lower bound for the Higgs boson mass, one finds

$$a_\mu(\text{weak}) = 195(1)\times 10^{-11}, \tag{3}$$

in the second order of perturbation theory,[6] which is only a factor of 5 smaller than the present experimental error. This means that if measurement of a_μ is improved by an order of magnitude, a_μ will provide an important testing ground of gauge theories of the electroweak interaction at the one-loop level, independent of processes such as muon decay, Cabibbo universality, $|\Delta S| = 1$ semileptonic decays of neutral particles, K_L-K_S mass difference, and mass shifts of W and Z bosons, which also require one-loop corrections for good fits.[7]

In order to realize such a test, however, it is necessary to improve the theoretical error in (2) by an order of magnitude. Most of this error ($\sim 9\times 10^{-9}$) comes from the hadronic vacuum polarization effect shown in Fig. 1(a). This contribution can be written as

$$a_\mu(\text{had}^{(a)}) = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{4m_\pi^2}^{\infty} \frac{ds}{s^2} K(s) R(s), \tag{4}$$

where $K(s)$ is a slowly varying function of s close to 1 except near the $\pi\pi$ threshold, and

$$R(s) = \sigma_{\text{tot}}(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-). \tag{5}$$

The error in (4) reflects directly the uncertainty in the measurement of R. Because of experimental difficulties in cutting down this error, the prevailing view has been that the CERN result (1) is

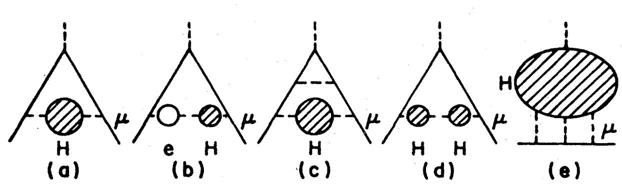

FIG. 1. Examples of hadronic contributions to a_μ. Photons are represented by dotted lines. Symbols e and μ refer to electron and muon lines. The symbol H means hadronic vacuum polarization.

the last word[8] and better measurement of a_μ is not worth the effort.

The purpose of this Letter is to show that the situation is not nearly that desperate. During the last two years we have improved the precision of the hadronic vacuum polarization contribution using the most recent measurements of R, reevaluated the contribution of diagrams containing hadronic light-by-light subdiagrams,[9] and also evaluated for the first time the complete α^4 QED contribution to a_μ.[10] The new theoretical value, including the tauon vacuum polarization effect (4.2×10^{-10}), the weak interaction effect (3), as well as the improved α^3 QED term, is

$$a_\mu{}^{\rm th} = 11\,659\,202(20)\times10^{-10}, \tag{6}$$

which is almost 5 times as accurate as (2), although this error must be regarded with some caution as is explained later. If the difference between (1) and (6) is due to compositeness of the muon and is of order $(m_\mu/M)^2$, where M is the mass of constituent particles, then M will be of order 1 TeV, which is 50% higher than the previous estimate.[1] Lower bounds on the masses of the supersymmetric partners of the leptons[2] will be improved similarly.

Whether the error of (6) is literally correct or not, the message that we should like to convey is that the theoretical error is now down to a size comparable to the magnitude of the weak interaction effect (3), bringing the latter within the range of laboratory detection. We believe that the error of (6) can be reduced further, in particular, in view of the novel approach to the measurement of R at CERN[11] which detects a $\pi^+\pi^-$ pair produced by a 300-GeV e^+ (from π^0 decay) incident on the electrons of target atoms. In this experiment, in which $\pi^+\pi^-$ and $\mu^+\mu^-$ pairs are counted simultaneously, $R(s)$ can be measured with an absolute accuracy of a few percent. The same experiment using the 1-TeV proton beam at Fermilab will extend the range to $s^{1/2}\sim 1$ GeV. The theoretical error of a_μ will eventually go down to 3×10^{-10} or less, removing a major obstacle for the experimental test of the electroweak effect (3). Thus, it now appears to be the opportune time to launch a new measurement of a_μ designed to reduce the error of (1) by at least an order of magnitude.[12]

In the following we shall outline the main features of our calculations. Details will be published elsewhere.[9,10]

α^4 QED contribution.—The old theoretical value (2) includes an estimate[4] of the α^4 term $[135(64)(\alpha/\pi)^4]$ obtained by a renormalization-group consideration,[13] which enables us to determine the coefficients of $\ln(m_\mu/m_e)$ terms without actual integration. Since this method is not capable of determining the mass-independent terms, the error quoted is nothing more than an educated guess. This error being of the same order of magnitude as the weak interaction effect (3), however, it is clearly necessary for our purpose to evaluate the α^4 term directly.

We have therefore carried out evaluation of the complete α^4 QED contribution to $a_\mu - a_e$ (which comes from 469 Feynman diagrams), slightly modifying the program written for computation of the electron anomaly a_e.[14] Including the reevaluated α^3 term,[15] our result can be written as[10]

$$a_\mu({\rm QED}) = 0.5(\alpha/\pi) + 0.765\,858\,10(10)(\alpha/\pi)^2 + 24.073(11)(\alpha/\pi)^3 + 140(6)(\alpha/\pi)^4 = 11\,658\,480(3)\times10^{-10}, \tag{7}$$

where we used the ac Josephson value of α[16]:

$$\alpha^{-1} = 137.035\,963(15). \tag{8}$$

Note that the value of the coefficient of $(\alpha/\pi)^2$, which includes the contributions of the electron and tauon vacuum polarization loops, and its error reflect the latest measurement[17] of m_μ. Although the use of this value, rather than the older one,[4] does not affect our result (6) for a_μ at present, this updating is made for the sake of future reference.[18]

Hadronic contributions.—Making use of the up-to-date measurements[19] of R, including that of Ref. 11, we have reevaluated the hadronic vacuum polarization contribution (4) with the result[9,20]

$$a_\mu({\rm had}^{(a)}) = 707(6)(17)\times10^{-10}, \tag{9}$$

where the first error is statistical and the second is systematic. Note that the former is more than 10 times less than the previous errors.[21,22] The latter is mostly due to the systematic error of the $\pi^+\pi^-$ channel. Better measurements of R in the low-energy range ($\sqrt{s}\lesssim 2$ GeV) are needed to improve the situation substantially.

As for the higher-order hadronic contributions to a_μ arising from the diagrams of Figs. 1(b)–1(d), the previous estimates[22] are adequate for the moment:

$$a_\mu({\rm had}^{(b)}) = 110(14)\times10^{-11},$$
$$a_\mu({\rm had}^{(c)}) = -207(29)\times10^{-11}, \tag{10}$$
$$a_\mu({\rm had}^{(d)}) = 2\times10^{-11}.$$

The contribution of the hadronic light-by-light subdiagram of Fig. 1(e) was calculated previously[22] under the assumption that it can be approximated by u, d, s, and c quark loops with $m_d = m_u = 0.3$ GeV, $m_s = 0.5$ GeV, $m_c = 1.5$ GeV. In view of the large error in the reported result $[-26(10) \times 10^{-10}]$, we have reevaluated it in two ways. One is under the same assumption as in Ref. 22 (except that the expansion in m_μ/m_q is not made), and the other is by approximating the diagram of Fig. 1(e) by a charged pion loop and various low-energy resonances (of which the π^0 resonance is the most important). The results are[9]

$$a_\mu(\mathrm{had}^{(e)}) = \begin{cases} 60(4)\times 10^{-11} \text{ (quark loop)}, & (11) \\ 49(5)\times 10^{-11} \text{ (pion loop and resonances)}, & (12) \end{cases}$$

which are consistent with each other but disagree strongly with the previous evaluation. We believe that the disagreement is due to poor convergence of the numerical integration in Ref. 22. Summing up the results (9), (10), and (12), we find the total hadronic contribution to be[23]

$$a_\mu(\mathrm{had}) = 702(19)\times 10^{-10}, \qquad (13)$$

where we have combined statistical and systematic errors for simplicity.

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