Universal Behavior of Sinai Billiard Systems in the Small-Scatterer Limit

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Sinai billiard systems are studied numerically and analytically. Let h be the Kolmogorov-Sinai entropy, $\langle\tau\rangle$ the mean free time of the point mass, and ϵ the scaling factor of the size of convex scatters (so that $\epsilon\to 0$ implies vanishing scatters). The following universal behavior is conjectured for *any* periodic Lorentz model in $d(\geq 2)$-space:

$\lim_{\epsilon\to 0} h\langle\tau\rangle/(-\ln\epsilon) = d.$

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Sinai billiard systems[1] or periodic Lorentz gas models[2] have been studied for a long time to clarify the logical foundation of equilibrium and nonequilibrium statistical mechanics. These systems have been studied extensively by mathematicians, but there are only a very few quantitative results known. The simplest Sinai billiard system consists of a point mass and a square table with a circular obstacle of radius R on it. The point mass moves without friction and is elastically reflected by the boundary of the obstacle and, upon hitting the edges of the table, disappears to reappear on the opposite side with the same velocity (periodic boundary condition). Thus we may think that the point mass moves geodesically on the two-torus (T^2), occasionally elastically scattered by the obstacle. When we consider this problem in the covering space of T^2, we call this the Lorentz gas problem on the square lattice with circular obstacles.

Sinai's celebrated theorem[1] tells us that as long as $R>0$, the system is chaotic, or more precisely, the system is a K system,[3] one of the most chaotic dynamical systems, and that the Kolmogorov-Sinai (KS) entropy[4] h is positive. Roughly speaking, the KS entropy measures the rate of loss of information contained in the ensemble of initial conditions; if the beam of trajectories of a billiard initially subtending a small angle ϵ spreads over the angle $\phi(t)$ after t sec, the rate of loss of information is $\sim[\ln\epsilon^{-1} - \ln\varphi(t)^{-1}]/t$. Thus the KS entropy is a good measure of "disorder" of dynamical systems; a more precise but still intuitive explanation can be found in the work of Oono and co-workers.[5] When $R=0$, i.e., there is no scatterer, it is obvious that the system cannot be chaotic; the system cannot even be ergodic. Thus there is an "order-disorder" transition in the $R\to 0$ limit.

As a part of our quantitative study of Sinai billiard systems, in this Letter, we study the simplest problem, i.e., the transition (or the critical) behavior from the chaotic to nonchaotic regimes. Our result can be summarized in the following conjecture in the simplest square lattice case. Asymptotically for small R, we have

$$\langle\tau\rangle h \sim -2\ln R, \tag{1}$$

where $\langle\tau\rangle$ is the mean free time. (Throughout this paper, $A\sim B$ implies A/B converges to unity in the appropriate limit.) It is not difficult to show intuitively that $\langle\tau\rangle h$ is of order $-\ln R$ (actually, a proof is given below). At each collision, the angle subtended by the beam of trajectories is magnified by $\sim R^{-1}$. Hence $\phi(t)$ in the preceding paragraph behaves like $\sim\epsilon(R^{-1})^{t/\langle\tau\rangle}$. Hence $h \sim -\ln R/\langle\tau\rangle$. What we claim in (1) is that the proportionality constant is *exactly* 2 (i.e., the spatial dimensionality). Moreover, later we will discuss that (1) is virtually universal, independent of the lattice structure and the number and shapes of convex scatterers.

Since between two collisions with the scatterer the particle travels along a straight line with a constant velocity which we normalize to unity, if we specify all the collision data, we can describe the system completely. Each collision can be described by the angle φ ($\in[\pi/2, 3\pi/2]$) of incidence and the position r $\{\in[0, 2\pi R)\}$ of the collision on the boundary of the scatterer (see Fig. 1). Thus the map T which describes the relation between the coordinates for the nth and $(n+1)$st collisions contains all the information about the dynamics; the relation between T and the original continuous dynamical system is given by the so-called Ambrose-Kakutani representation.[6] Moreover, the KS entropy h of the original system can be calculated from the KS entropy h_T of the T map according to the Abramov formula[7]

$$h = h_T/\langle\tau\rangle. \tag{2}$$

An explicit formula for h_T can be found in Kubo.[1] The formula can be understood intuitively by the "dilution ratio formula,"[5a] and reads

$$h_T = \int d\nu \ln\left(1 - \tau_1 \frac{\chi^{(e)}(r,\varphi) + R^{-1}}{\cos\varphi}\right), \tag{3}$$

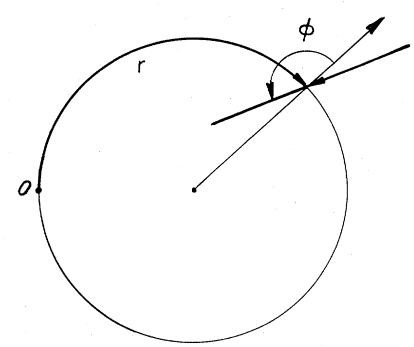

FIG. 1. Definition of variables used in the T map.

where $d\nu = -(4\pi R)^{-1}\cos\varphi\, d\varphi\, dr$, τ_n is the free time between the $(n-1)$st and the nth collisions,

$$\chi^{(e)}(r,\varphi) = R^{-1} - \cos\varphi \left[\tau_0 + \cfrac{1}{-2R^{-1}/\cos\varphi_{-1} + \cfrac{1}{\tau_{-1} + \cdots \; \ddots \; \tau_{-n} + \cfrac{1}{-2R^{-1}/\cos\varphi_{-n} + \cdots}}}\right], \tag{4}$$

and $T^n\varphi = \varphi_n$.

We have calculated h_T numerically by three methods. In the first method we work directly from formula (3). We approximate the function $\chi^{(e)}(r,\varphi)$ by truncating the partial fraction expansion. The phase average is performed as a time average. In the second method, we compute the entropy via the Pesin formula[8] $h_T = \int \lambda_>(x) d\nu$, where $\lambda_>(x)$ is the maximum characteristic exponent at the phase point x. In our case $\lambda_>(x)$ is a constant and so $h_T = \lambda_>$. $\lambda_>$ is calculated as $N^{-1}\ln\|DT^N\vec{v}\|$, where $\vec{v}$ is an arbitrary normalized vector, DT is given by

$$DT = \begin{pmatrix} \partial r/\partial r_1 & \partial r/\partial\varphi_1 \\ \partial\varphi/\partial r_1 & \partial\varphi/\partial\varphi_1 \end{pmatrix},$$

and N has been chosen suitably large. This is essentially the same method used by Benettin and Strelcyn[9] to compute the entropy of the stadium. In the final method we use the expression $h_T = \int \mu\, d\nu$, where μ is the largest characteristic value of DT. We perform the phase average as a time average.

All three methods appear to give the same value of the entropy for each R ($< \frac{1}{2}$). Thus it appears that the conclusion of Pesin's theorem is correct in our case though the assumption for it is not satisfied. The results are given for the square-lattice and the triangular-lattice Lorentz-gas[10] cases in Fig. 2. Our numerical asymptotic relation for the T-map entropy is

$$h_T \sim (-2 \pm 0.02)\ln R, \quad R \lesssim 0.05, \tag{5}$$

where 0.02 is a conservative estimate of the consistency of the numerical data. Note that the curve of h_T vs R for the square case is translated upward by roughly $\ln(2/\sqrt{3})$ from the similar curve in the triangular case. This is reasonable since $\sqrt{3}/2$ is the ratio of the areas of unit cells of both cases.

Analytically, we can proceed as follows. The continued fraction in (4) contains only positive fractions, so that

$$0 \leq \frac{1}{-2R^{-1}/\cos\varphi_{-1} + 1/(\tau_{-1} + \cdots)} \leq \frac{1}{-2R^{-1}/\cos\varphi_{-1}} = -\frac{R}{2}\cos\varphi_{-1} \leq \frac{R}{2}.$$

Thus h_T can be written as

$$h_T = (4\pi R)^{-1}\int_0^{2\pi R} dr \int_{\pi/2}^{3\pi/2} d\varphi \ln\left(1 + \frac{\tau_1}{\tau + \theta R} + \frac{2\tau_1}{R\cos\varphi}\right)(-\cos\varphi),$$

where θ is a function such that $|\theta| \leq \frac{1}{2}$. We can rewrite this into

$$h_T = (4\pi R)^{-1}\int_0^{2\pi R} dr \int_{\pi/2}^{3\pi/2} d\varphi \ln\left(\frac{2\tau}{R\cos\varphi}\right)(-\cos\varphi) + I(R) = \langle \ln\tau/R\rangle + 1 + I(R). \tag{6}$$

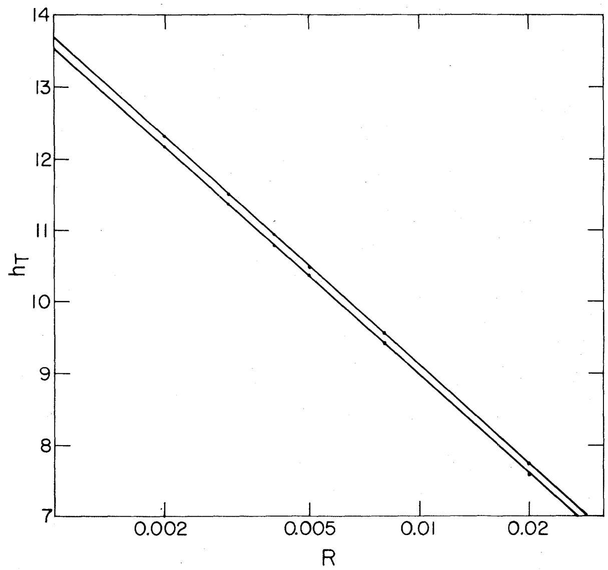

FIG. 2. Entropy of the T map vs the radius of the obstacle. Upper curve is for the square lattice. Lower curve is for the triangular lattice. In both cases, the lattice spacing is always chosen to be 1.

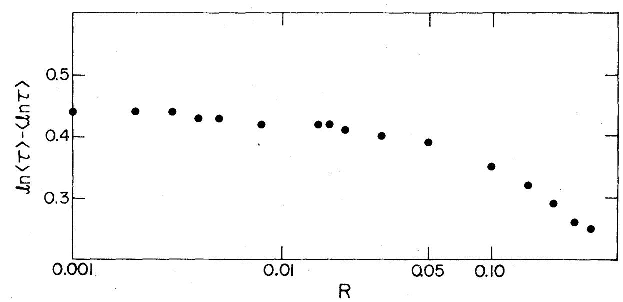

FIG. 3. Graph of $\ln\langle\tau\rangle - \langle\ln\tau\rangle$ vs R, the radius of the scatterer. We estimate (from the consistency of the numerical data) the error in $\ln\langle\tau\rangle - \langle\ln\tau\rangle$ to be ± 0.01.

We can show that $I(R) \sim \pi R/4$ in the limit $R \to 0$. We must estimate $\langle\ln\tau\rangle$. Since we can calculate $\langle\tau\rangle = -\pi R/2 = 1/2R$, we have

$$\langle\ln\tau\rangle \leq \ln\langle\tau\rangle \leq \ln(1/2R) \sim -\ln R.$$

It is not difficult to show that

$$-[2-\epsilon(R)]^{-1}\ln R < \langle\ln\tau\rangle.$$

Thus in the limit $R \to 0$, we have the following rigorous bounds:

$$-\tfrac{3}{2}\ln R < h_T = \langle\tau\rangle h \leq -2\ln R + \text{const}. \qquad (7)$$

Hence $\langle\tau\rangle h$ is of order $\ln R$ in the limit $R \to 0$. (5) and (7) suggest that asymptotically

$$h_T \sim -2\ln R. \qquad (8)$$

Since our billiard is without finite horizon, i.e., τ is not bounded from above, we can have very long free paths. Actually, the mean square free time $\langle\tau^2\rangle$ is not finite. Thus there is definitely a long-time tail in the velocity correlation function. Its origin is easy to understand: very long free paths. Despite the existence of these long-time tails, since $\langle\tau\rangle \sim 1/2R$ $(R \to 0)$, a "mean-field" type argument can give a correct estimate of $\langle\tau\rangle$. This suggests that for τ^α $(\alpha \leq 1)$ the contribution of the long-time tail can be ignored. Thus we can conjecture

$$\lim_{R\to 0}\langle\ln\tau\rangle/\ln\langle\tau\rangle = 1, \qquad (9)$$

checked in the triangular lattice case. Actually, our numerical results suggest a far stronger conjecture:

$$\lim_{R\to 0}(\langle\ln\tau\rangle - \ln\langle\tau\rangle) = \text{const}. \qquad (10)$$

(see Fig. 3), which, of course, implies (9). (6) can be shown even if there are many circular scatterers of radius R. Thus we may conjecture that (1) is true in this case as well.

For d-dimensional cases we can show an analogous formula to (6) asserting $h_T \sim \langle\ln\tau/R\rangle$ if the scatterers are all identical d spheres of radius R. Again we may expect that $\langle\ln\tau\rangle \sim \ln\langle\tau\rangle$ and that $\langle\tau\rangle$ is of order R^{1-d} $(R \to 0)$. Hence $h_T \sim -d\ln R$, i.e., we may conjecture that

$$\lim_{R\to 0}\langle\tau\rangle h/(-\ln R) = d. \qquad (11)$$

We can easily generalize this conjecture to cover the most general case with arbitrary convex scatterers. This can be done by introducing a scaling factor ϵ of the size of scatterers. If we consider the uniform shrinkage of scatterers by this factor, then in the limit $\epsilon \to 0$ we have (7) with R replaced by ϵ.

For the triangular billiard, for $R < \sqrt{3}/4$ the maximum free time is unbounded, while for larger values of R there is a finite horizon. We have examined numerically the entropy of the T map around $R = \sqrt{3}/4$. Our study shows that the entropy is continuous and suggests that it is even continuously differentiable in a neighborhood of $R = \sqrt{3}/4$.

The main message of the conjecture is that we may completely ignore the fluctuation of the free time to estimate the leading order of divergence of h_T in the small-scatterer limit even though there is no finite horizon and long-time tails exist. The situation is drastically different when we try to calculate $\langle\tau^\alpha\rangle$ for $2 > \alpha > 1$.

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