Suppression of Period Doubling in Symmetric Systems

James W. Swift and Kurt Wiesenfeld
Department of Physics, University of California, Berkeley, California 94720
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The role of symmetry is examined in systems displaying period-doubling instabilities. It is found that symmetric orbits will not undergo period doubling except in extraordinary cases. Such exceptional cases cannot occur in a large class of systems, including the sinusoidally driven damped pendulum and the Lorenz equations.

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A common instability found in nonlinear dynamical systems is the period doubling of a periodic orbit.[1,2] As some parameter is smoothly varied past a critical value, the limit cycle loses stability to an orbit having twice this period. When the limit cycle possesses a symmetry, an interesting phenomenon is observed. A number of digital and analog investigations have found that symmetric periodic orbits will first bifurcate to nonsymmetrical orbits before a period doubling can occur.[3-10]

The *necessity* of a symmetry-breaking precursor to period doubling has been asserted either implicitly or explicitly by several investigators.[4-7] The first explanation attempted in the physics literature appears to be the discussion by D'Humieres *et al.*[5] Those authors restructed their attention to the sinusoidally driven pendulum, and based their results on the stability analysis of an explicit (though approximate) solution of the governing equation. More recently,[11] a heuristic explanation was offered, based solely on the symmetry of the governing equation and the bifurcating limit cycle. It is the purpose of this Letter to elucidate the role of symmetry in suppressing period-doubling instabilities. Our discussion proceeds most naturally in the context of bifurcation theory for mappings.[12-14]

Before introducing the technical details of our analysis, let us state the main result of our work. Broadly put, we conclude that a symmetric periodic orbit *can* directly bifurcate to a period-double orbit; however, this can only occur under exceptional circumstances. In particular, we prove that such exceptional cases *cannot* arise in a large class of systems, which includes the driven damped pendulum.

We begin by defining the symmetry under consideration. Consider the periodically driven systen (with driving period T) described by an n-dimensional system of ordinary differential equations:

$$\dot{x} = F(x,t); \quad x \in R^n, \quad F(x, t+T) = F(x,t). \tag{1}$$

We call the system (1) *symmetric* if F satisfies

$$F(x,t) = -F(-x, t+T/2). \tag{2}$$

An example of a symmetric system is the driven, damped pendulum, which is governed by Eq. (1) with

$$x = \begin{pmatrix} \theta \\ \omega \end{pmatrix}; \qquad F(x,t) = \begin{pmatrix} \omega \\ -\kappa\omega + U'(\theta) + A\cos(2\pi t/T) \end{pmatrix}. \tag{3}$$

Here, κ is the damping constant, $U(\theta) = \cos\theta$, and A is the driving amplitude.

A general consequence of the symmetry (2) is that if $x^*(t)$ is a solution, then so is $-x^*(t+T/2)$. We call $x^*(t)$ a *symmetric solution* of period T if $x^*(t) = -x^*(t+T/2)$.

Given any initial condition $x(t_0)$, Eq. (1) may be integrated to find x at some later time t_1. In this way, the governing equation (1) defines a solution map

$$P_{t_0}{}^{t_1}: R^n \to R^n; \quad x(t_0) \to x(t_1).$$

Clearly, this map has the property

$$P_{t_0}{}^{t_2} = P_{t_1}{}^{t_2} \circ P_{t_0}{}^{t_1}, \tag{4}$$

where the open circle denotes composition of mappings. For periodically driven systems with driving period T, the map

$$P = P_{t_0}{}^{t_0+T} \tag{5}$$

is called the Poincaré map. Without loss of generality, we set $t_0 = 0$ for the remainder of this Letter.

We now make a crucial observation, upon which all the results of this Letter hinge. *For symmetric systems, the Poincaré map is the second iterate of another map.* To see this, observe that the symmetry (2) may be expressed as

$$x \to -Ix, \quad t \to t+T/2, \tag{6}$$

where I is the $n \times n$ unit matrix. Symmetry of the solution map $P_{t_1}{}^{t_2}$ implies that it commutes with the symmetry (6), that is

$$(-I)\circ P_{t_1}{}^{t_2} = P_{t_1+T/2}{}^{t_2+T/2}\circ(-I). \tag{7}$$

From (4) and (5), we have

$$P = P_{T/2}{}^{T}\circ P_0{}^{T/2} = (-I\circ P_0{}^{T/2}\circ -I)\circ P_0{}^{T/2}, \tag{8}$$

this last step following from (7). The maps are associative, so that

$$P = (-I\circ P_0{}^{T/2})^2 \equiv (\tilde{P})^2,$$

verifying the italicized statement above. The situation is depicted in Fig. 1.

Note that any T-periodic orbit of (1) corresponds to a fixed point of P. In contrast, only a symmetric orbit of (1) corresponds to a fixed point of $\tilde{P}$, while a nonsymmetric orbit yields a two-cycle of $\tilde{P}$.

Other investigators have recognized that the Poincaré map may be decomposed in this way for specific driven and autonomous symmetric systems.[7,15,16]

We now study the bifurcations of symmetric orbits. To use the results of bifurcation theory,[12,14] we must have a mapping with no special properties. Therefore $\tilde{P}$, not P, is the correct map to use for symmetric orbits. We will assume in what follows that $\tilde{P}$ has no further symmetries and that $\tilde{P}$ cannot itself be expressed as the square of another map.

A bifurcation occurs when the fixed point x_0 of $\tilde{P}$ loses stability. If x_0 is stable, then points x near the fixed point approach x_0 under iterations of $\tilde{P}$. Whether this is the case is determined by studying the map $\tilde{P}$ linearized about x_0:

$$x \to D\tilde{P}(x_0)X. \tag{9}$$

If the eigenvalues $\tilde{\mu}_i$ of the matrix $D\tilde{P}(x_0)$ all have modulus less than 1 then x_0 is asymptotically stable. As the parameters of the system are varied, the eigenvalues move around in the complex plane. A bifurcation is signaled when at least one eigenvalue exits from the unit circle. The stability of the orbit may also be found by examining the eigenvalues μ_i of $DP(x_0)$. For symmetric orbits, $DP(x_0) = [D\tilde{P}(x_0)]^2$, so that $\mu_i = (\tilde{\mu}_i)^2$, whereas for nonsymmetric orbits $DP(x_0)$ is not the square of another matrix, and an analogous relation does not hold. It follows that bifurcations of nonsymmetric orbits are studied via the full map P.

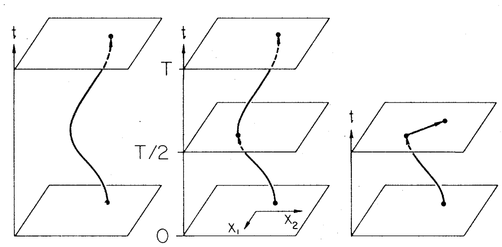

FIG. 1. (a) $P = P_0{}^T$; (b) $P = P_{T/2}{}^T\circ P_0{}^{T/2}$; (c) $P = -I\circ P_0{}^{T/2}$.

In what ways may a symmetric orbit lose stability? This depends crucially on the nonlinear terms neglected by the linear map (9). The classification scheme of bifurcation theory tells us that if only a *single parameter* is varied, then the bifurcation will almost surely be one of the following three types[12,14] (these are the codimension-one bifurcations of the mapping $\tilde{P}$):

(i) *Saddle node.*—A single eigenvalue exits from the unit circle at $+1$. Two symmetric limit cycles (i.e., fixed points of $\tilde{P}$) collide and annihilate. When the parameter exceeds the critical value, there are no longer any fixed points in the neighborhood of x_0, and the solution rapidly evolves to a different region of phase space. In experiments this is reflected by a dramatic jump in the system's response.

(ii) *Symmetry breaking.* (period doubling of $\tilde{P}$).—A single eigenvalue exits from the unit circle on the negative real axis. This gives rise to a two-cycle of $\tilde{P}$. As mentioned above, this two-cycle corresponds to a nonsymmetric period-T limit cycle.

(iii) *Hopf.*—A pair of complex conjugate eigenvalues $(\tilde{\mu}, \tilde{\mu}^*)$ crosses the unit circle. Under the assumption that the eigenvalues satisfy a nonresonance condition ($\tilde{\mu}^n \neq 1$, $n = 1$–4), there is an invariant torus created (or annihilated) at the bifurcation.

When two parameters are varied, many other possibilities exist. Among them is the resonant case mentioned above, where $\tilde{\mu} = \pm i$. This bifurcation is of special interest because x_0 can lose stability to a stable four-cycle of $\tilde{P}$, i.e., $\tilde{P}$ undergoes *period quadrupling*,[12] and thus P undergoes period doubling. Let us clarify why period quadrupling of $\tilde{P}$ is considered exceptional if it occurs in a one-parameter system, but unexceptional if it occurs in a two-parameter system. The point is that there exist infinitesimal perturbations of the former which restore the nonresonance condition $\tilde{\mu}^4 \neq 1$. In contrast, for the case of two-parameter systems, all nearby two-parameter families will have critical parameter values where $\tilde{\mu} = \pm i$.

From the foregoing considerations, we conclude that a symmetric periodic orbit *may* undergo period doubling as a single parameter is varied through some critical parameter value, but that this is a highly unlikely prospect.

In fact, for certain systems we can say much more: We can guarantee that $\tilde{P}$ cannot undergo period quadrupling even if there are two or more parameters. For this purpose, it is convenient to focus on the eigenvalues μ_i of $DP(x_0)$. Since period quadrupling of $\tilde{P}$ corresponds to the pair $\tilde{\mu} = \pm i$, it follows that $DP(x_0)$ must have two eigenvalues simultaneously cross the unit circle at -1. Specifically, we consider the case where the Poincaré mapping is two dimensional [$n = 2$ in Eq. (1)], so that there are precisely two eigenvalues. A general theorem from Floquet theory[17] fixes the product of all the eigenvalues to be

$$\prod_i \mu_i = \exp\{\int_0^T \operatorname{div} F(x^*(t))\,dt\}. \tag{10}$$

To understand this relation, note that the left-hand side of (10) gives the factor by which volumes of phase space expand or contract under the map P. The right-hand side is obtained by integrating the change in the infinitesimal comoving volume

$$\frac{1}{\text{volume}}\,\frac{d(\text{volume})}{dt} = \operatorname{div} F(x^*(t))$$

over one period of the orbit.

It follows that if

$$\int_0^T \operatorname{div} F(x^*(t))\,dt < 0 \tag{11}$$

and the Poincaré section is two dimensional, the symmetric orbit $x^*(t)$ cannot undergo period doubling since this would require $\mu_1\mu_2 = 1$ at the bifurcation.

Of course, (11) is sure to hold when $\operatorname{div} F(x)$ is negative for all x. For example, consider the class of systems (3) describing the motion of a damped, driven oscillator in any symmetric potential $U(\theta)$. We have

$$\operatorname{div} F = \partial\dot{\theta}/\partial\theta + \partial\dot{\omega}/\partial\omega = -\kappa,$$

with κ a constant. Note that the driving term does not contribute to the divergence, although it does feed energy into the system.

We may interpret these results in a very physical way: if P is two dimensional and both eigenvalues are equal to -1, then $DP(x_0)$ is an area-preserving map—but this is impossible in a purely dissipative system [i.e., when $\operatorname{div} F(x) < 0$ for all x].

In fact, *any* Hopf bifurcation ($\mu_1 = \mu_2^* = e^{i\alpha}$) of symmetric or nonsymmetric orbits is impossible in dissipative systems with two-dimensional Poincaré maps, since $\mu_1\mu_2 < 1$ for such systems.

We close with three remarks. First, the experimental signal of a symmetric orbit is the presence of only *odd* multiples of the fundamental frequency. This follows directly from the definition $x^*(t + T/2) = -x^*(t)$. At the symmetry-breaking bifurcation the even harmonics appear. Second, one can define symmetric period-$(2n + 1)T$ orbits by

$$X^*(t + (n + \tfrac{1}{2})T) = -X^*(t).$$

These orbits are $(2n+1)$-cycles of $\tilde{P}$. The results of this Letter apply to *any* symmetric orbit. In particular, symmetric orbits are often observed to undergo the symmetry-breaking bifurcation, followed by the period-doubling cascade and chaos. This occurs when a fixed point of $\tilde{P}$ (or $\tilde{P}^{2n+1}$) follows the sequence of bifurcations familiar from one-dimensional maps.[1,2] Note that the stable three-cycle of $\tilde{P}$ in the "window" of the chaotic region is a symmetric period-$3T$ limit cycle. This orbit itself undergoes the symmetry-breaking bifurcation before the period-doubling cascade.

Finally, our results can be extended in various ways, the most important being to autonomous (i.e., time-independent) systems. In addition, modes with the symmetry $y \to y$ can be present together with modes obeying the symmetry $x \to -x$. In these systems, as well as (1) and (2), the Poincaré map *of a symmetric orbit* is the square of another map. Thus, our analysis can be applied (for instance) to the model equations describing doubly diffusive convection.[18-20] The symmetry-breaking bifurcation has been observed in numerical studies of convection.[3,4,7,8,10] Furthermore, one can prove that symmetric orbits of purely dissipative three-dimensional autonomous systems—such as the Lorenz equations[7,21]—cannot directly undergo a period-doubling bifurcation.

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