Crises and Hysteresis in Coupled Logistic Maps

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A new type of "crisis" is found in coupled logistic maps. In this crisis several chaotic attractors undergo cyclic collisions with their basin boundaries and merge into a bigger one, whose chaotic dynamical behavior exhibits cyclic transitions among its components with a well defined overall period.

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The sudden qualitative change in chaotic dynamical behavior induced by the collision of a strange attractor with an unstable fixed point has been called crisis of the chaotic attractor by Grebogi, Ott, and Yorke.[1,2] Crises of two kinds have been identified: (i) a boundary crisis, where a strange attractor collides with an unstable fixed point on the boundary of the basin of attraction, causing the disappearence of both basin and chaotic attractor; and (ii) an interior crisis, where the collision occurs within the basin of attraction, and produces almost always a sudden expansion of the attractor itself.

In this paper, we present evidence for the existence of a new type of crisis, characterized by the simultaneous cyclic collision of several coexisting chaotic attractors with the boundaries that separate the respective basins of attraction. This situation is illustrated schematically in Fig. 1. Immediately after collision, as the control parameter is continuously varied, all the attractors that take part in the cyclic collision merge into a single larger attractor. Its dynamical behavior is characterized by cyclic transitions of the phase-space trajectory from one to the other of the original attractors in a fixed order, but with seemingly random residence times on each of them.

Examples of this type of crisis are provided by the symmetric two-dimensional logistic map with

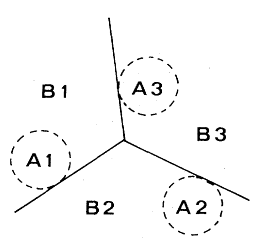

FIG. 1. Illustration of a cyclic crisis encountered by three chaotic attractors (A1, A2, A3) with separate basins of attraction (B1, B2, B3).

a bilinear coupling term:

$$x_{n+1} = F(x_n, y_n) = 4\lambda x_n(1-x_n) + \gamma x_n y_n, \quad y_{n+1} = F(y_n, x_n) = 4\lambda y_n(1-y_n) + \gamma y_n x_n. \tag{1}$$

A mapping of the type given by Eqs. (1) and the corresponding one with a linear coupling term[3-5] have been shown to exhibit complicated dynamical behavior, including quasiperiodicity, phase locking, intermittency, period adding, long-lived chaotic transients, etc. In this paper, we concentrate only on the dynamical behavior associated with the crisis of Eqs. (1) by selecting $\gamma = 0.1$ and varying λ. In addition to the cyclic crisis mentioned above, we have also found evidence for a boundary crisis of a type similar (but not identical) to the one found in Henon's map by Grebogi, Ott, and Yorke.[6] A boundary crisis occurs in our case through the collision of a chaotic attractor with the basin boundaries that separate it from several other coexistent periodic or quasiperiodic attractors (another chaotic attractor, in their case). Upon an increase of λ beyond its critical value for the onset of a crisis, the chaotic attractor and its basin disappear while the basins of the remaining attractors undergo a sudden expansion. This, in turn, produces hysteresis effects[7] in the dynamics as λ decreases through its critical value.

The Lyapunov exponents provide a convenient way to characterize quantitatively the thresholds of the various dynamical behaviors and to measure the randomness of the chaotic, quasiperiodic, and periodic attractors. In particular, with the help of the largest two Lyapunov exponents, we have traced out the route to chaos of a single attractor by selecting as the starting point of a given iteration the final condition corresponding to the previous values of λ. On observing a sudden change in dynamical behavior, as revealed by the Lyapunov exponents or by the phase protrait, we have reversed the direction of variation of λ and searched for the existence of hysteretic behavior.

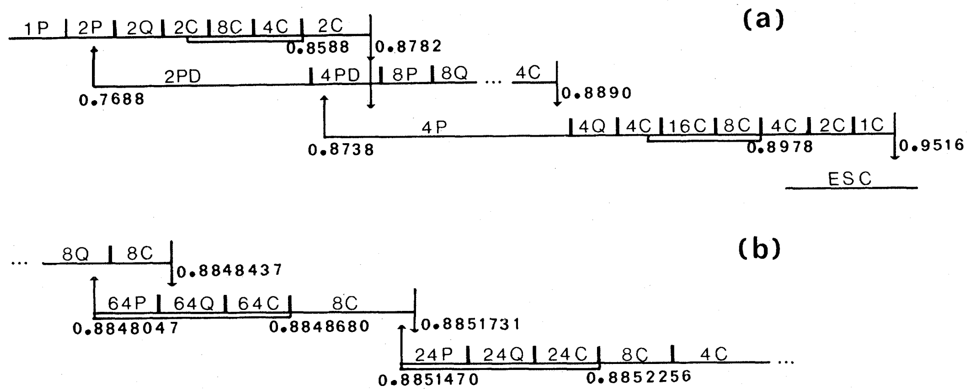

FIG. 2. Bifurcation scheme of the map (1) with $\gamma = 0.1$. The attractors are denoted by P (periodic), PD (periodic attractor located on the diagonal), Q (quasiperiodic), C (chaotic), and ESC (escape to infinity). The integer n that preceeds the symbols indicates that the attractor can be divided at most into n separate parts such that each part is an attractor of the map $F^{(n)}$ $[F^{(n)}(x,y) \equiv F(F^{(n-1)}(x,y),\ F^{(n-1)}(y,x)),\ F^{(1)} \equiv F]$. We use multiple horizontal lines to represent several coexisting attractors with their own basins. Double lines under an attractor symbol denote coexisting attractors which are mirror images of each other. Frequency locking and windows within the quasiperiodic and chaotic regions are not indicated.

With this procedure, we have obtained the stability domain in the space of the control parameter λ (see Fig. 2).[8] As shown, in particular, in Fig. 2(a), the entire stability domain can be separated into three main branches, each with its own bifurcation route to chaos. There are clear similarities between the bifurcation routes of the upper and lower branches. The middle branch, on the contrary, contains more complicated substructures as shown in Fig. 2(b). Details of the various bifurcation schemes will be discussed elsewhere. Here we focus, instead, on the nine cases of crises resulting from this map.

These nine crises can be divided into two groups: The first (boundary crises) consists of five cases which are indicated in Fig. 2 by downward arrows. As an example, we discuss in some detail the crisis occurring at $\lambda_{C1} \simeq 0.8782$. Figure 2(a) shows the existence of a region with $\lambda \lesssim \lambda_{C1}$ where three attractors coexist (these are labeled 2C, 4PD, and 4P following a nomenclature that is explained in the caption to Fig. 2). We have identified their respective basins of attraction and displayed them in Fig. 3. Also shown in this figure is the phase portrait of the 2C attractor which almost touches its basin boundary. We note that the 2C attractor seems to approach its basin boundaries with the 4P and with the 4PD attractors at the same time. Above the threshold value λ_{C1} the 2C attractor and its basin suddenly disappear while the basins of the 4P and 4PD expand to fill the domain previously occupied by the 2C basin. This expansion occurs in a highly interlaced fashion.[9] The best way to illustrate this point is to examine the behavior of the maximal Lyapunov exponent (MLE) in the range $0.84 < \lambda < 0.9$, as shown in Fig. 4. The MLE's in this case are all calculated by selecting a fixed initial point for the iteration ($x_0 = 0.3$, $y_0 = 0.4$) which is located in the 2C basin when $\lambda = \lambda_{C1}$. Our results show that, after the destruction of the 2C attractor, the plot of the Lyapunov exponent appears to split into two curves. A closer examination shows that the values of the MLE's undergo frequent jumps as λ varies in this region. This

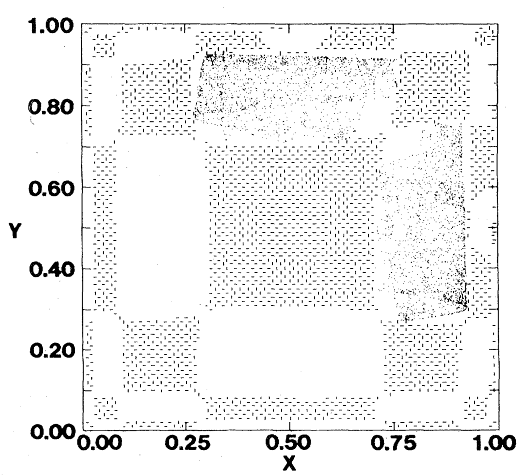

FIG. 3. Basin structure and 2C attractor for $\gamma = 0.1$, and $\lambda = 0.8782$. The blank region represents the basin of 2C; vertical lines denote the basin of 4P; horizontal lines the basin of 4PD.

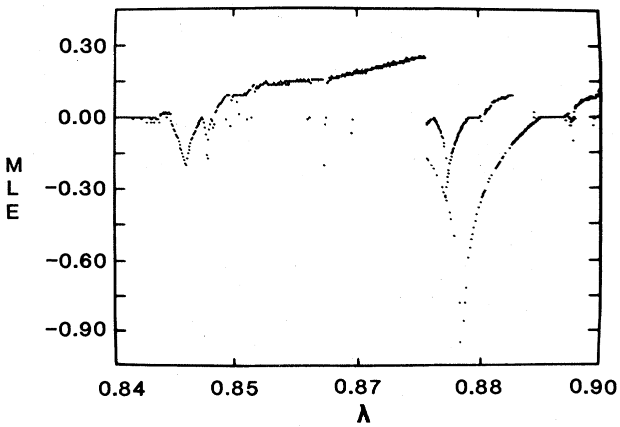

FIG. 4. Maximal Lyapunov exponents for $\gamma=0.1$ and λ in the range $0.84 \lesssim \lambda \lesssim 0.9$. The initial conditions are fixed at $x_0=0.3, y_0=0.4$.

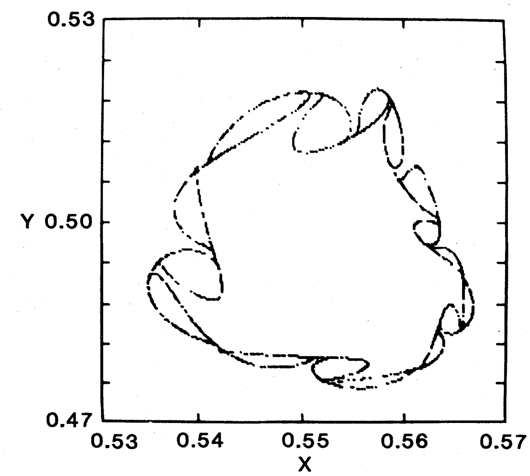

FIG. 5. One of the eight chaotic attractors of the map $F^{(64)}$ at $\gamma=0.1$ and $\lambda=0.884\,868\,3$.

phenomenon arises from the highly interlaced nature of the 4P and 4PD basin boundaries which move past the point (x_0,y_0) many times upon slight variations of the control parameter λ. On reversal of the direction of the variation of λ through λ_C and tracing of the path of either the 4PD or 4P attractors, the system does not return to the upper branch as shown in Fig. 2(a); we conclude that hysteresis is always associated with such crises.

The remaining four crises which are located at the right ends of the horizontal double lines in Fig. 2 belong to the second category (cyclic crises) as mentioned in the introductory paragraphs.[10] In all these four cases, and when λ is smaller than, but close to, the critical crisis value, we always find two coexistent chaotic attractors which are mirror images of each other. To be more specific, we shall focus on the crisis occurring at $\lambda_{C2} \simeq 0.884\,868\,0$ which can be found on the middle subbranch of Fig. 2(b). The entire interval corresponding to the horizontal double lines consists of two coexisting attractors which are mirror images of each other. This interval begins with two 64P attractors which bifurcate via a Hopf bifurcation into two 64Q attractors. The quasiperiodic attractors then evolve into two sets of 64 chaotic islands (64C) through frequency locking and intermittency. Perhaps the best way to describe this crisis is to consider the $F^{(64)}(x_n,y_n)$ map. ($F^{(n)}$ is defined in the caption of Fig. 2.) When λ is slightly smaller than λ_{C2}, $F^{(64)}$ generates 128 chaotic attractors, each with its own basin of attraction. At the threshold value, $\lambda=\lambda_{C2}$, sets of sixteen chaotic attractors coalesce into one, leading to only eight final attractors. The collisions between the sixteen attractors and the respective basin boundaries take place simultaneously in a cyclic fashion as illustrated in Fig. 1. An example of the outcome of these collisions is shown in Fig. 5, which displays sixteen rings linked together. Immediately after the crisis ($\lambda \gtrsim \lambda_{C2}$) the system trajectory of the $F^{(64)}$ map moves around the sixteen linked rings in a cyclic order. This is illustrated in Fig. 6 by the temporal record of the y coordinate for one of the eight chaotic attractors. The residence time on each of the rings appears to be random, but the travel time (or period) T required to go through a complete cycle, when averaged over many cycles, is a constant that

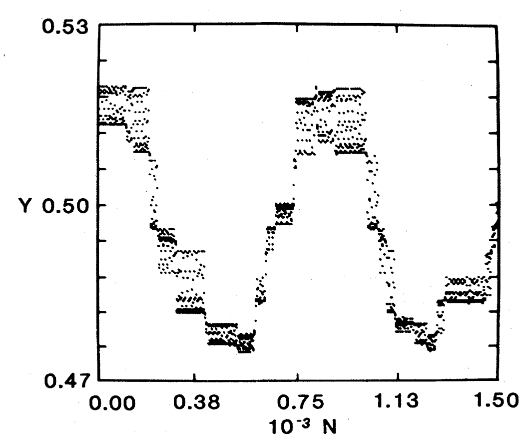

FIG. 6. Time record of the y coordinate of the attractor shown in Fig. 5. The horizontal axis is labeled by the number of iterations of $F^{(64)}$.

does not seem to depend on the initial conditions. We find empirically that the averaged period $\langle T\rangle$ scales as $(\lambda - \lambda_C)^{-k}$ in the vicinity of a cyclic crisis. More specifically, around λ_{C2} the k value is approximately 0.53 which is probably a consequence of the fact that the overlapping region between the original attractor and the basin of the other attractor is a one-dimensional curve.[6,11] In contrast, around $\lambda_{C3} \simeq 0.8588$, k is close to 1.0, corresponding to a two-dimensional region of overlap. These results are consistent with our observation that only the maximal Lyapunov exponent is positive for the chaotic attractor around λ_{C2}, but both the largest two Lyapunov exponents are positive around λ_{C3}. As λ approaches λ_{C2} from above, the average period increases indefinitely, and the system trajectory settles down in one of the sixteen attractors when λ becomes smaller than λ_{C2}. No hysteresis exists for a cyclic crisis.

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[1]C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. 48, 1507 (1982).

[2]C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. 50, 935 (1983).

[3]K. Keneko, Prog. Theor. Phys. 69, 1427 (1983).

[4]J. M. Yuan, M. Tung, D. H. Feng, and L. M. Narducci, Phys. Rev. A 28, 1662 (1983).

[5]T. Hogg and B. A. Huberman, "Generic Behavior of Coupled Oscillations" (to be published).

[6]C. Grebogi, E. Ott, and J. A. Yorke, Physica (Utrecht) 7D, 181 (1983).

[7]Y. Yamaguchi and K. Sakai, Phys. Rev. A 27, 2755 (1983).

[8]An initial point on the diagonal necessarily reduces Eq. (1) into a small one-dimensional map, whose attractors may not be stable against small perturbations in a two-dimensional space. Only stable kinds are entered in Fig. 2.

[9]Chaotic transients always exist for this kind of crisis. This is intimately related to the highly interlaced nature of the basin structure.

[10]Cyclic crises have also been observed for cases where the two λ's in Eqs. (1) are taken to be different. Their occurrence in these cases can be related to the existence of a chaotic transient.

[11]We thank Professor J. A. Yorke for drawing our attention to the relation between k and the dimension of a chaotic attractor.