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Stochasticity and Transport in Hamiltonian Systems

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The theory of transport in nonlinear dynamics is developed in terms of "leaky" barriers which remain when invariant tori are destroyed. A critical exponent for transport times across destroyed tori is obtained which explains numerical results of Chirikov. The combined effects of many destroyed tori lead to power-law decay of correlations observed in many computations.

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With the proof of the Kolmogorov-Arnol'd-Moser (KAM) theorem,[1] our knowledge of the regular motion in Hamiltonian systems was given a firm foundation. An integrable Hamiltonian system with N degrees of freedom has N invariants (such as energy, momentum, etc.), which restrict motion in the $2N$-dimensional phase space to surfaces which are N-dimensional tori. When a small enough perturbation is added to an integrable Hamiltonian, the KAM theorem asserts that most of the invariant tori will still exist. Complementary results from numerical experiments lead one to describe motion in regions of phase space where tori are destroyed as irregular or stochastic. The sense in which the stochastic motion can be described as random has, however, proved elusive. In this Letter we report the beginning of a detailed description of stochasticity in two-degree-of-freedom systems.[2]

It is convenient to use the surface-of-section method to reduce the Hamiltonian flow to an area-preserving map. The intersection of an invariant torus with a surface of section is topologically a circle, which we will refer to simply as a circle. Orbits on the torus wind around helically and will repeatedly pierce the surface at points on the circle which rotate with some average frequency ν. To be definite take a radial coordinate p and an angular coordinate x (with period 1) and represent the surface of section map by $(x_1, p_1) = T(x_0, p_0)$ where $T(x+1, p) = T(x, p)$. A rotational invariant circle of frequency ν is a continuous curve parametrized by t with $-\infty < t < \infty$ such that $(x(t+\nu), p(t+\nu)) = T(x(t), p(t))$ and $(x(t+1), p(t+1)) = (x(t)+1, p(t))$.

Let us follow an invariant circle with given frequency as a perturbation with magnitude governed by a parameter k is added to the map. If the frequency is "sufficiently irrational" (far from low-order rationals) the KAM theorem implies that the invariant circle will exist at finite values of the parameter k. Quite generally we expect that invariant circles persist up to some critical value of the parameter, $k_c(\nu)$. It is typically zero for rational ν. Above k_c an invariant circle with rational frequency $\nu = m/n$ is replaced by a chain of n islands, while one with irrational frequency develops gaps.[3] In the latter case the existence of a single gap implies that there are an infinity of gaps since the iterates of the end points of a gap rotate with an irrational frequency. Since a set, like the destroyed circle, consisting of a curve deleting a dense set of open intervals is a Cantor set, the remnant of the invariant torus is called a cantorus. In Fig. 1 a cantorus for the standard or Chirikov-Taylor map[4] is displayed.

Motion in the neighborhood of a cantorus appears stochastic and conversely stochastic re-

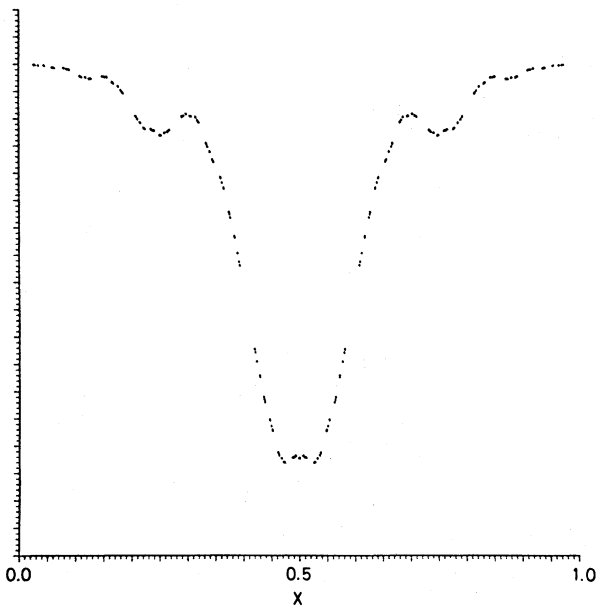

FIG. 1. A cantorus of the standard map with frequency $(1+\sqrt{5})/2$ for a parameter value 0.03 above critical in polar coordinates.

gions appear to be stratified by these remnants of destroyed circles. Since the motion on the cantorus has a definite rotation frequency, a point near a cantorus will for some finite time appear to rotate with the same frequency. However, the cantori are unstable invariant sets so that nearby orbits will eventually diverge from them. The importance of the cantori in stochastic motion and a transport theory to describe the flow through stochastic regions are the primary results of this Letter.

We believe that cantori are the principal impediments to transport, and present numerical evidence supporting this in Ref. 2. Recall that a torus partitions three-dimensional space into two regions, and thus an invariant torus forms an absolute barrier to the flow. The cantorus provides a leaky barrier: Any orbit which escapes from one region must go through the gaps.

The flux through the gaps may be visualized by constructing the leaky barrier from the stable and unstable manifolds of the cantorus. Focus attention on one of the largest gaps in the cantorus. Since the total length of the gaps must be finite, the iterates of this gap as $t\to\infty$ must have widths which approach zero. Area preservation implies that this longitudinal stability of the end points of the gap must be accompanied by a transverse instability. The future invariant, or stable, manifold of the cantorus may be constructed by

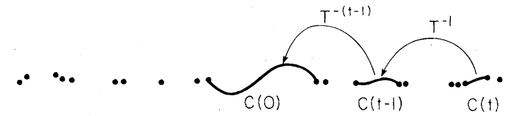

FIG. 2. Construction of the stable manifold of a cantorus by interating backwards from a far future gap.

joining the end points of the tth iterate of the large gap with a curve $C(t)$ and iterating back to $t=0$ (Fig. 2) to obtain a curve $C(0)$ in the large gap. As the iteration time goes to infinity $C(0)$ approaches an invariant form $C^+ = \lim_{t\to\infty} T^{-t}C(t)$ which, because of the longitudinal stretching, is a smooth curve connecting the end points of the large gap. A point on C^+ approaches the cantorus in the future. The unstable manifold, C^-, is constructed similarly with use of a curve through a far past iterate of the large gap, noting that the width must go to zero as $t\to-\infty$ as well. If the cantorus has only a single family of gaps (all gaps are iterates of any one), then the positive iterates of C^+ and negative iterates of C^- then form a curve which closes all the gaps except that it has two pieces in the gap at $t=0$. Were C^+ and C^- to coincide, this curve would be an invariant circle with rotation number ν, but this contradicts the assumption $k>k_c(\nu)$. These curves must therefore be different and enclose an area (Fig. 3). We call this structure the turnstile. In some cases there may be more than one family of gaps and then each has its own turnstile. We have constructed the stable and unstable manifolds for the standard map obtaining a structure like that in Fig. 3.

In Fig. 3 C^+ and C^- cross at precisely one point (this may, however, not be true in general) which is homoclinic to the end points of the gap: It approaches the cantorus in both directions of time. The area enclosed by each lobe of these manifolds, ΔW, represents the flux through the cantorus—the area which crosses the cantorus per iteration. This follows because a point in the left lobe (for example) of the turnstile is both inside the unstable manifold and outside the stable manifold, so that a past iterate is inside the barrier

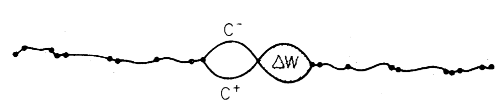

FIG. 3. Sketch of the turnstile in the gap at $t=0$, and partial barrier constructed from the stable and unstable manifolds.

while a future iterate is outside. The area of the right half of the turnstile gives, correspondingly, the inward flux.

Thus ΔW is like a local transport coefficient: It is the flux of trajectories across a destroyed invariant circle. It can be shown[2] that ΔW is related to an action, which is defined as the integral of the Lagrangian along an orbit. ΔW is the difference between the action of the orbits formed from a gap end point and that of the homoclinic point. Mather has used precisely this quantity in a proof of the existence of invariant circles.[5] He shows that $\Delta W \geq 0$ and when $\Delta W = 0$ an invariant circle exists. Our construction has provided a physical interpretation for ΔW.

The most important cantori will be those that are slightly supercritical and for which ΔW is still small. This is just the case when Hamiltonian systems have universal scaling properties[6] describing motion in the neighborhood of the cantorus. Application[2] of this analysis yields a critical exponent for ΔW:

$$\Delta W \propto [k - k_c(\nu)]^{\eta}, \quad \eta \simeq 3.011\,722,$$

for the case when ν is a "noble" number. Noble irrationals are those which are most difficult to approximate by rationals, and consequently appear to be most resistant to perturbation[5]—that is, $k_c(\nu)$ is locally maximum when ν is noble. The fact that ΔW grows so slowly with k explains why it is extremely difficult to detect the breaking of invariant circles by looking for trajectories which cross the cantorus[4]: As an example 10^8 iterations of a simple map like the standard map gives only 1% accuracy in k_c.[2] The exponent η can be compared with numerical experiments of Chirikov[4] for the transition time from one region to another. Chirikov measures that this time goes to infinity with an exponent −2.55 while our theory gives $-\eta$ since the time is proportional to the inverse of the flux. The difference between these results is probably due to Chirikov's use of data far from critical to obtain a fit to power-law scaling.

A global picture of transport is obtained by considering the combined effect of the barriers of all the cantori.[2] In the simplest description only those most resistant barriers with locally minimum ΔW are kept. If we label the regions bordered by two such cantori with integers, then ΔW_{ij} is the area of the turnstile in the cantorus between regions i and j. If the area of the stochastic component in region i is A_i then we can assume that the probability that an orbit lands in the turnstile is $\Delta W_{ij}/A_i$. If this is small then many iterations of the map will be necessary before an orbit crosses the barrier, giving an effective loss of memory, so that we can use a Markovian approximation with a transition probability per step $p(i \to j) = \Delta W_{ij}/A_i$.[2]

This picture of transport can be used to predict confinement times, e.g., for guiding centers in tokamaks. It also explains the long-time correlations seen in dynamical systems with regular and irregular regions. As a stochastic orbit approaches an invariant circle it must traverse barriers with $\Delta W_{ij} \to 0$. An orbit trapped in region i has a time constant roughly $\tau_c \sim A_i / \sum_j \Delta W_{ij}$; this time goes to infinity as the invariant circle is approached [for a critical noble circle $\tau_c \sim (2.618)^i$ where the regions A_i are chosen to represent successive continued-fraction approximations to the noble frequency]. This infinite sequence of time constants leads to a power-law decay of correlations similar to that observed in many systems.[7]

It is no more difficult to include the effect of island chains as well as the rotational cantori on transport. Any stable periodic orbit of period n will be surrounded by a chain of invariant circles, that is, circles invariant under T^n. Beyond the outermost circle will be chains of cantori as well. To include these cantori in the Markovian model a transition probability $p(i \to j)$ must be added for each, so that for each region there are branches corresponding to the hierarchy of islands around islands.

The theory of barriers and turnstiles has many implications in nonlinear dynamics and for stochasticity in dynamical systems. The construction of barriers provides a set of dynamical variables which approximate true action-angle variables. The action, W, of a particular barrier (the phase-space area contained on one side) has a natural uncertainty, ΔW, associated with its nonconservation. An angle variable can be defined which obeys $\theta' = \theta + \nu$ on a barrier, except when θ falls in the turnstile in which case ν changes as the trajectory is transported to a new barrier. No other choice of action variables would be better conserved, or equivalently have a smaller flux. The idea of minimizing flux goes back to Wigner[8] and similar ideas have been developed in parallel to ours by Bensimon and Kadanoff.[9] It may be possible to obtain a rigorous transport theory utilizing the approximate action-angle variables.

Another application is to semiclassical quantization. As it is usually formulated semiclassical theory is only defined when the classical tori exist. Since, however, quantum mechanics effectively averages over areas smaller than $\hbar$ one may be able to quantize using approximate tori formed from barriers with $\Delta W < \hbar$. Of course it may not usually happen that a cantorus at the appropriate value of the action ($W \sim n\hbar$) will have small enough ΔW, but this technique may at least extend semiclassical theory into the stochastic regime. Reinhardt and co-workers[10] have used the similar, but less precise, concept of "vague" tori in their semiclassical theory.

Finally we mention that in systems with more than two degrees of freedom, invariant tori do not partition the energy surface. Therefore, at first sight the cantori do not appear to be important in higher dimensions. It is interesting to note, however, that Arnol'd diffusion is calculated with use of Melnikov's integral[4] which measures the degree to which a separatrix is broken and has parallels with Mather's ΔW. Since in two degrees of freedom the flux across separatrices is larger than that across cantori, it could be that higher-dimensional cantori play an important role in Arnold diffusion.

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