PHYSICAL REVIEW LETTERS

VOLUME 52 | 13 FEBRUARY 1984 | NUMBER 7

Relativistic Perturbations of an Earth Satellite

Neil Ashby
Department of Physics, University of Colorado, Boulder, Colorado 80309

and

Bruno Bertotti
Dipartimento di Fisica Nucleare e Teorica, Università di Pavia, 27100 Pavia, Italy
(Received 6 June 1983)

The inertial frame of reference in the neighborhood of a test body provided by the construction of Fermi normal coordinates is generalized to include the effect of the body's gravitational field. The metric obtained provides a simple physical description of relativistic corrections to the orbital motion of a satellite of the Earth. The main correction is the nonlinear Schwarzschild field of the Earth; in these coordinates there are also three much smaller terms arising from the solar tidal influence.

PACS numbers: 04.20.Jb, 04.20.Me, 95.10.Eq

The intrinsic error in a number of operational laser ranging systems used to track satellites such as LAGEOS is now about 3 cm, and may be expected to decrease to about 1 cm in a few years, while modeling errors are currently about 25 cm.[1] Knowledge of higher harmonics in Earth's gravitational field will continue to improve, while uncertainties in nongravitational accelerations can be reduced with very dense satellites in high orbits, by compensation with drag-free systems, and by measurements with onboard accelerometers as well as by development of improved models for periodic drag and radiation-pressure effects. Thus residuals in fitting of models to precise tracking data may be expected to improve to the centimeter level in the next few years. Since Earth's gravitational radius is $\mu_e = GM_e/c^2 \approx 0.443$ cm, it will soon be essential to include fractional relativistic corrections of order μ_e/r in modeling precise tracking of Earth-orbiting spacecraft. A straightforward description of the relevant relativistic effects is clearly needed.[2]

In the case of interplanetary orbits, where these effects are solely due to the sun, this task is reduced to the study of the Schwarzschild solution and is well understood. For near-Earth satellites, where the potential of the sun and Earth are comparable, one should use the relativistic solution to the n-body problem.[3,4] The slow-motion, weak-field (SMWF) approximation[4,5] provides a general formalism for calculation of the required post-Newtonian corrections. But because of arbitrariness in the coordinates, as described by gauge freedom,[5] the equations of motion one obtains[6] do not readily lend themselves to physical interpretation. A comparison with observation can be made only by integrating such equations and constructing in an invariant manner the physically observed quantities. In the carrying out of this procedure large terms tend to cancel out and new terms arise. As a result, relativistic effects on the orbit of an Earth-orbiting satellite are not easily describable in physical terms and have been the subject of some controversy.[2]

We shall summarize here the results of work in which the motion of a test body in the neighborhood of Earth is described in a particularly convenient frame of reference. This frame is the generalization, in the SMWF approximation, of the local, inertial frame which can be constructed in the neighborhood of a geodesic in the field of the sun.[7] The generalization consists in including the field of Earth and allowing for nonlinear Earth-sun interactions. In these coordinates one must use expansion parameters different from that usually used in the SMWF approximation, $\epsilon = O(\mu/R)$, where μ is the gravitational radius of the sun and R its distance. The deviation from flatness of the metric due to the sun is measured instead by $\mu r^2/R^3$, with r the distance from the center of Earth.

The construction of this generalized Fermi metric is carried out in two steps. First, given a geodesic world-line G in the field due to the sun only, one builds on it a vierbein, $[\Lambda_{(0)}{}^{\mu}, \Lambda_{(i)}{}^{\mu}]$, $i=1,2,3$, and the corresponding Fermi normal coordinates (x^{μ}). The metric in the local inertial frame is obtained either by performing the coordinate transformation from Schwarzschild isotropic coordinates X^{μ} to x^{μ}, or by using its explicit expression in terms of the Riemann tensor.[5,7] If we denote the ordinary velocity along G by $V^i = dX^i/dX^0$, it is necessary to compute g_{00} up to terms of order $\mu V^2/r^2R^3$, while for the other components of the metric tensor, the lowest significant terms suffice. These are the expressions one gets:

$$g_{00}+1 \equiv h_{00}{}^S = 2U_T - (\mu/R^3)[r^2(6\mu/R - 4V^2)P_2 + V^2(r^2 - 3x_V{}^2) + 9x_V x_R VV_R - 3r^2V_R{}^2]; \tag{1a}$$

$$g_{0i} \equiv h_{0i}{}^S = -(2\mu/R^3)[x_i(x_RV_R - x_V V) + \hat{R}_i(2x_V x_R V - V_R r^2) + V_i(r^2 - 2x_R{}^2)]; \tag{1b}$$

$$g_{ij} - \delta_{ij} \equiv h_{ij}{}^S = -(\mu/3R^3)[\delta_{ij}(2r^2 - 3x_R{}^2) - 2x_i x_j - 3\hat{R}_i\hat{R}_j r^2 + 3(x_i\hat{R}_j + x_j\hat{R}_i)x_R]. \tag{1c}$$

Here,

$$U_T = 2\mu r^2 P_2/R^3 = \mu(3x_R{}^2 - r^2)/R^3 \tag{2}$$

is the ordinary tidal potential, and P_2 is a Legendre polynomial. Subscripts R and V denote, respectively, the components along the radius vector $\vec{R}$ from the sun to Earth, and Earth's velocity $\vec{V}$; $r = \sqrt{(\delta_{ij}x^i x^j)}$ is the Cartesian distance from G, and $\hat{R} = \vec{R}/R$. Geodetic precession affects the tidal potential and other terms in Eqs. (1a)–(1c) because the vierbein rotates with respect to the barycentric frame by about 0.019″/yr. The metric tensor (1) is correct in the limit $r/R \to 0$ for a point Earth; in the term $h_{00}{}^S$ the coefficient of $\mu r^2/R^3$ has been evaluated up to terms of order $V^2 \approx \mu/R$. This expression, therefore, is consistent only if r/R is no larger than $O(\mu/R)$; for larger values of r/R the expansion in r/R must be taken beyond quadratic terms. It is sufficient to do so, however, in the tidal potential U_T. This can easily be done in a Newtonian framework and in the following discussion we shall understand U_T to be the tidal potential suitably corrected to the necessary order in r/R.

The second step is application of the coordinate transformation $X^{\mu} \to x^{\mu}$ defined above to the full post-Newtonian metric[6,8] including Earth. The central line G is taken at the center of Earth. One needs g_{00} up to quadratic terms in the masses of Earth and the sun and the other components up to linear terms; this determines the required approximation in the functions $X^0(x^{\mu})$ and $X^i(x^{\mu})$. The calculation is complicated, but it reveals subtle corrections which arise at this high level of precision. In particular, second-order effects appearing to arise from the retardation of gravitational signals are cancelled by time-dilation and length-contraction effects so that relativistic perturbations on the acceleration of near-Earth satellites are reduced to a very small level.

We show here how to arrive at the final expression for the metric in a direct, albeit less rigorous, way. As far as h_{0i} and h_{ij} are concerned, one can simply add to the solar contributions (1b) and (1c) those due to Earth; since in the local frame Earth is at rest, the additional terms are obtained from the Schwarzschild solution in isotropic form. The only nonlinear terms needed are those in g_{00}. Among them, those coming from the sun only are already given by (1a); those due to Earth are obtained from the Schwarzschild solution; therefore only the interaction term, denoted by δh_{00}, is needed. Therefore we set

$$g_{00} = -1 + h_{00}{}^S + 2U - 2\mu_e{}^2/r^2 + \delta h_{00}, \tag{3a}$$

$$g_{0i} = h_{0i}{}^S, \tag{3b}$$

$$g_{ij} = \delta_{ij} + h_{ij}{}^S + \delta_{ij} 2\mu_e/r. \tag{3c}$$

In Eq. (3a), the ordinary Newtonian potential U differs from μ_e/r by the contributions due to the multipole terms of Earth's field and the other bodies in the solar system except the sun itself; these contributions are not needed in Eq. (3c) because in the equations of motion, h_{ij} appears only in the correction to the Newtonian acceleration. δh_{00} is determined by the field equation with two time indices; when written in terms of $h_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu}$ to within terms of second order it reads

$$\nabla^2 h_{00}=h^{ij}h_{00,ij}+h_{00,i}h^{ij}{}_{,j}-\tfrac{1}{2}h_{00,i}h_{00}{}^{,i}-\tfrac{1}{2}h_{00,i}h^{j}{}_{j}{}^{,i}. \qquad (4)$$

The comma means ordinary derivative and the indices are raised with Kronecker's δ_{ij}; Latin indices run from 1 to 3. We need to solve this equation only as far as the terms proportional to $\mu\mu_e$ are concerned; they arise from terms on the right side of (4), of the type $Uh_{\mu\nu}{}^S$. In $h_{\mu\nu}{}^S$ it is sufficient to consider the terms of order $\mu r^2/R^3$. A straightforward calculation then gives

$$\nabla^2\delta h_{00}=\tfrac{40}{3}(\mu\mu_e/rR^3)P_2. \qquad (5)$$

By requiring that δh_{00} be of $O(r)$ one obtains the unique solution

$$\delta h_{00}=-\frac{10}{3}\frac{\mu r^2}{R^3}P_2\frac{\mu_e}{r}. \qquad (6)$$

It is remarkable that the solution is proportional to the product of U and the tidal potential U_T, as one could have expected in a naive, nonlinear theory; the coefficient $\frac{10}{3}$ is of course peculiar to general relativity.

We next discuss the invariance properties of this solution. Just as in the Fermi solution,[9] we can replace the time x^0 with an arbitrary combination x^0+c and operate on x^i with an arbitrary and constant orthogonal transformation. We have also some freedom in the choice of G. Suppose that the center of Earth is not placed at the origin, but at a point $\vec{r}_e$, and that its velocity in this frame is not zero, but $d\vec{r}_e/dt=\vec{v}_e$ with $v_e\approx Vr_e/R$. Then Earth's potential is not μ_e/r, but the time-dependent function $\mu_e/|\vec{r}-\vec{r}_e|$. This brings about a correction in δh_{00} and in the higher-order gravity tensor for the earth. The field equation (4) has now a correction, coming from $h_{00,00}$, of order

$$\frac{\partial}{\partial x^0}\frac{\mu_e}{|\vec{r}-\vec{r}_e|^2}v_e\approx\frac{\mu_e}{r^2}\frac{\mu r_e}{R^3}$$

(in fact the acceleration of Earth is now $d\vec{v}_e/dt\approx\mu r_e/R^3$). This is negligible with respect to the source term in (5) if $r_e\ll r$. It can also be shown that with this condition the full Earth field remains essentially unchanged. Therefore, the solution (3) is invariant under a group of transformations which has the same structure as Poincaré's group, although the translations and the boosts must be small in the above sense.

It is straightforward to write the Lagrange function corresponding to the metric (3). We confine ourselves to listing the different types of terms which arise. We have

$$L=\underbrace{\tfrac{1}{2}v^2-U-U_T}_{\{0\}}+\underbrace{\frac{\mu_e^2}{r^2}+v^2\frac{\mu_e}{r}+v^4}_{\{1\}}+\underbrace{\frac{\mu}{R^3}\frac{\mu}{R}r^2}_{\{2\}}+\underbrace{\frac{\mu}{R^3}vVr^2}_{\{3\}}+\underbrace{\frac{\mu}{R^3}[\mu_e r+v^2r^2]}_{\{4\}}. \qquad (7)$$

In order, the terms in the above equation are the classical Lagrangian {0}; the relativistic Schwarzschild corrections to the field of Earth, {1}; a correction to the tidal field of the sun which is quadratic in its mass, {2}; a "magnetic" term linear in the velocity of the test body, arising because in this

TABLE I. Estimates of perturbing accelerations of an Earth satellite due to relativistic corrections. In evaluating these orders of magnitude it is assumed that $v^2\simeq\mu_e/r$. For LAGEOS, $r/(10^9\text{ cm})\simeq1.2$. The magnitudes of the accelerations due to Newtonian forces of attraction to Earth, and the solar tides, are given for comparison.

Source	Magnitude
Newtonian potential	$c^2\mu_e/r^2=4\times10^2[(10^9\text{ cm})/r]^2$ cm sec^{-2}
Solar tides	$c^2\mu r/R^3=4\times10^{-5}[r/(10^9\text{ cm})]^2$ cm sec^{-2}
(1) Nonlinear Earth field	$c^2\mu_e{}^2/r^3=2\times10^{-7}[(10^9\text{ cm})/r]^3$ cm sec^{-2}
(2) Nonlinear solar tides	$c^2\mu^2r/R^4=4\times10^{-13}[r/(10^9\text{ cm})]$ cm sec^{-2}
(3) Solar "magnetic" term	$c^2[\mu^3\mu_e r/R^7]^{1/2}=8\times10^{-13}\sqrt{[r/(10^9\text{ cm})]}$ cm sec^{-2}
(4) Earth-sun interaction	$c^2\mu\mu_e/R^3=2\times10^{-14}$ cm sec^{-2}

frame the sun is in motion, {3}; and interaction between the sun's tidal field and the Newtonian potential of Earth, {4} (an analogous nonlinear interaction with the lunar tidal field is to be expected but is not discussed here). In Table I are listed the orders of magnitude of the corresponding accelerations.

The independent space and time variables defined by the above construction are the obvious extension, in general relativity, of Earth-centered inertial coordinates and Newtonian time. They provide a convenient and readily interpretable reference frame for the description of precision tracking of Earth satellites and for worldwide clock synchronization.[10,11] For such applications, one important reason why the relativistic corrections of Table I are so small is that the equations of transformation from barycentric coordinates to local inertial coordinates require clocks to be synchronized in the local inertial frame. Such clocks will not be synchronous with respect to barycentric coordinate time.

We conclude that the main relativistic effects upon an Earth satellite are those described by the Schwarzschild field of Earth itself. It is well known that the only secular perturbation from Earth's field is an advance of the perigee, $\delta\omega$, of order μ_e/r radians per orbit plus a similar change in the mean motion. Since the actual observable for advance of perigee is not $\delta\omega$ but $\delta\omega$ times the eccentricity, for a satellite in an almost circular orbit like LAGEOS this effect is just now becoming observable. The new relativistic effects, corrections to the tidal field of the sun, are much smaller and have a different dependence upon the distance. They are below the present level of observability for LAGEOS, but the first term (No. 2 in the table) may be barely observable with the laser ranging to the moon. We shall discuss in detail these three corrections in a future paper.

One of us (B.B.) received partial support through a fellowship at the Joint Institute of Laboratory Astrophysics University of Colorado, Boulder, Colorado, and also through the Consiglio Nazionale delle Richerche.

[1]B. E. Schutz, B. C. Tapley, and R. J. Eanes, in "Error Sources in SLR Baseline Determinations," Third Crustal Dynamics Working Group Meeting, Goddard Space Flight Center, October 28–29, 1982 (unpublished).

[2]C. F. Martin and M. H. Torrence, Eos 62, 261(A) (1981).

[3]W. de Sitter, Mon. Not. Roy. Astron. Soc. 76, 699 (1916), and 77, 155 (1916).

[4]Arthur Eddington and G. L. Clark, Proc. Roy. Soc. London, Ser. A 166, 465 (1938).

[5]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), pp. 1069 ff.

[6]Theodore D. Moyer, Cel. Mech. 23, 33, 57 (1981).

[7]F. K. Manasse and and C. W. Misner, J. Math. Phys. 4, 735 (1963).

[8]C. M. Will and K. Nordtvedt, Jr., Astrophys. J. 177, 757–774 (1972).

[9]E. Fermi, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat. Rend. 31, 21, 51 (1922).

[10]M. Fujimoto, S. Aoki, K. Nakajima, T. Fukushima, and S. Matsuzaka, in *Geodetic Applications of Radio Interferometry, Proceedings of Symposium No. 5, General Meeting of the International Association of Geodesy, Tokyo, 7–8 May 1982*, National Oceanographic and Atmospheric Administration Report No. NOS 95 NGS24 (U.S. Dept. of Commerce, Washington, D.C., 1982), pp. 26–35.

[11]N. Ashby and D. W. Allan, Radio Sci. 14, 649 (1979).