Fractal Dimension and Self-Similarity of the Devil's Staircase in a Josephson-Junction Simulator

In a recent Letter, Jensen, Bak, and Bohr[1] showed by numerical calculations that the stability intervals for limit cycles of the circle map form a complete devil's staircase at the onset of chaos, and they conjectured that the fractal dimension of the Cantor set derived from the space between mode-locked plateaus should assume a universal value of $D=0.87$. This fractal dimension may serve as an index for characterizing the scaling behavior of a large class of systems at the critical point.

The system we used to measure this exponent was an electronic Josephson-junction simulator, the same as the one used in previous studies of chaos.[2] The essential property of this nonlinear system is described by the differential equation

$$\frac{d^2\varphi}{dt^2}+\frac{1}{(\beta_c)^{1/2}}\frac{d\varphi}{dt}+\sin\varphi=A_0+A_1\sin\omega t, \qquad (1)$$

where $\beta_c=2eI_cR^2C/\hbar$ is the McCumber parameter and the terms on the right-hand side of Eq. (1) are dc and ac currents in the circuit expressed in reduced units.[2] We have measured the widths of mode-locked dc current plateaus corresponding to rational winding numbers $W=f_J/f_d$, where f_J and f_d are the Josephson and driving frequencies, respectively. The stability intervals are determined by observation of mode-locked patterns of $\sin\varphi$ vs $\dot{\varphi}$ on an oscilloscope and by use of a spectrum analyzer to find the frequency ratio W while the dc current is varied.

For low values of β_c (e.g., $\beta_c=0.03$, and with $W=0.5495$, $A_1=1.473$), we were able to observe a large number of subharmonic plateaus in excess of a hundred. This lends strong support to the predicted behavior of *self-similarity* in the mode-locked structure, or a devil's staircase.[1,3] However, for this range of β_c, which is far from chaos, a plot of $\log N(r)$ vs $\log(1/r)$ yields a slope $D\cong 1$, where $N(r)=[1-S(r)]/r$, and $S(r)$ is the total length of subharmonic steps having a width larger than r in the closed interval of dc current variation, indicating that the devil's staircase is far from being complete.

With $\beta_c\cong 0.3$, and $\omega=1.6-2.9$, $A_1=0.6-1.1$, the system is in a state not far from chaos; the $\log N(r)$-$\log(1/r)$ plot then shows a *fractal dimension* with $D<1$. A typical plot is shown in Fig. 1. For several sets of selected parameters near the onset of chaos, the slope determined from this plot is $D=0.91\pm 0.04$. Intrinsic noise and stability problems have prevented us from measuring systematic variations of D near chaos with reliable accuracy. Also, since the system can enter chaos via various routes by changing several different parameters, and the onset of chaos through varying β_c is not sharply defined, it is difficult to assess quantitatively the deviation from the critical point. The value $D=0.91\pm 0.04$ thus represents the lowest value of the fractal dimension which has so far been observed with a narrow range of parameter variation near transition to chaos in our system. The change from $D\cong 1$ to 0.91 as the system approaches chaos, and the smooth straight lines shown in Fig. 1, can be viewed as supporting evidence for fractal dimension and self-similarity of the devil's staircase suggested by the numerical calculations of Jensen, Bak, and Bohr.

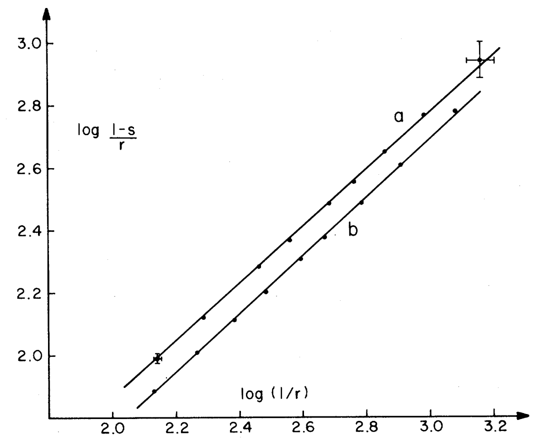

FIG. 1. Plots of $\log[1-S(r)]/r$ vs $\log 1/r$, slopes give the fractal dimension D. Curve a: $\omega=2.90$, $A_1=1.06$, A_0 ranges from $W=\frac{1}{8}$ to $\frac{1}{5}$, $D=0.91$, based on 66 subharmonic steps. Curve b: $\omega=1.58$, $A_1=0.63$, A_0 ranges from $W=\frac{1}{5}$ to $\frac{1}{3}$, $D=0.93$, based on 45 subharmonic steps.

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