Solution of the Multichannel Kondo Problem

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The multichannel Kondo model is exactly diagonalized for any impurity spin and for an arbitrary number of orbital channels. The impurity free energy is found and its properties deduced for high and low temperatures. When the number of channels is sufficiently large a nontrivial fixed point appears. Its critical exponents are calculated.

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The multichannel Kondo model

$$H=-i\sum_{a,m}\int dx\,\psi_{a,m}^*(x)\partial_x\psi_{a,m}(x)+2J\sum_{a,b,m}\psi_{a,m}^*(0)\vec{\sigma}_{ab}\psi_{b,m}(0)\vec{S} \qquad (1)$$

arises in the study of magnetic impurities in a metal when the orbital structure of the impurity atom is taken into account.[1] The particular form (1) applies when the crystal electric fields are larger than the spin-orbit coupling and split off an isotropic orbital singlet[1] (transition-metal impurities with half-filled levels).

The field $\psi_{a,m}$ describes electrons with spin index $a=\pm\frac{1}{2}$ and "flavor" index $m=1,\ldots,f$ (which labels the orbital channels), interacting, via spin exchange, with a "flavorless" spin-S impurity located at $x=0$. The infrared properties of the model, we shall find, depend crucially on the values of f and S. The original argument is due to Nozières and Blandin.[1] While the weak-coupling fixed point ($J=0$) is always unstable, the nature of the strong-coupling trivial fixed point ($J=\infty$) depends on the strength of the flavor interaction. For $f\leqslant 2S$ it is attractive and the effective coupling constant flows smoothly from weak to strong coupling, leading to a system characterized by an effective spin $S'=S-\frac{1}{2}f$ at $T=0$. In particular, if $S=\frac{1}{2}f$ the impurity spin is totally screened and one has a complete Kondo effect. A new behavior occurs if $f>2S$. In this case the impurity-induced flavor interaction is sufficiently strong to destabilize the strong-coupling fixed point. As a result a nontrivial infrared fixed point appears, which controls the low-temperature properties of the model.[1] The low-temperature regime is therefore expected to exhibit scaling behavior characterized by nontrivial critical exponents. We shall be able to determine these exponents in what follows.

To do so we shall diagonalize the Hamiltonian and derive an expression for the impurity free energy. The main difficulty is to identify and incorporate the effect of the flavor degrees of freedom. The first quantized form of the Hamiltonian is

$$h=\sum_{j=1}^{fN}[i\partial_j+2J\delta(x_j)\vec{\sigma}_j\cdot\vec{S}],$$

and flavor seems to have disappeared. This, of course, is not the case since it will make its appearance through the Pauli principle satisfied by the electron fields.

To see how it comes about consider a stream of electrons carrying spin and flavor degrees of freedom, all moving with the same velocity, $v_F=1$, and impinging on the impurity. Since the impurity is "flavor blind," as long as the electrons pass the impurity one at a time, the flavor will play no role. It enters by allowing more than one electron to be at the impurity site and interact with it. Since all electrons move with the same speed, this means that flavor allows composites

$$\Psi_{a_1\ldots a_n,\,m_1\ldots m_n}^*(x)=\psi_{m_1a_1}^*(x)\psi_{m_2a_2}^*(x)\cdots\psi_{m_na_n}(x),\quad n\leqslant f,$$

to form and interact with the impurity. The appropriate composite will be determined by the dynamics.

To observe the formation of these dynamic composites one must adopt a cutoff scheme that is sufficiently general.[2] Consider adding to the Hamiltonian higher-derivative terms, $(1/\Lambda)^{k-1}\times\psi_{am}^*(\partial_x)^k\psi_{am}$. These would provide a "built in" cutoff for the model.[3] To preserve factorizability one also must add local counter terms which are, however, irrelevant in the continuum limit, $\Lambda\to\infty$, and will not be specified here.[4]

We choose the second-order regulator and study the Hamiltonian

$$h=\sum_{j=1}^{fN}[-i\partial_j-\Lambda^{-1}(\partial_j)^2+2J\delta(x_j)\vec{\sigma}_j\cdot\vec{S}] \qquad (2)$$

by means of Bethe-*Ansatz* techniques. The eigenfunctions are given as combinations of plane waves with pseudomenta $\{k_j, j=1,\ldots,fN\}$ and spin-flavor-dependent coefficients. These are determined from the two-body S matrices derived from h,

$$S_{jl}=S_{lj}^{-1}=\frac{\lambda_j-\lambda_l-iJP_{jl}^{\text{spin}}}{\lambda_j-\lambda_l-iJ}\,\frac{\lambda_j-\lambda_l-iJP_{jl}^{\text{flavor}}}{\lambda_j-\lambda_l-iJ},\quad j,l\leq fN;$$
$$S_{j,fN+1}=S_{fN+1,j}^{-1}=\frac{\lambda_j+1-iJ(\vec{\sigma}_j\cdot\vec{S}+\frac{1}{2})}{\lambda_j+1-iJ(S+\frac{1}{2})},\quad j=1,\ldots,fN. \tag{3}$$

The quantities λ_j are given by $\lambda_j=k_j/\Lambda$ in our cutoff scheme. The spectrum is found by solving the periodic boundary conditions and is given by

$$E=\sum_{j=1}^{fN}k_j(1+k_j/\Lambda),$$

where the momenta (all distinct by Fermi statistics) are derived from[5]

$$\exp(ik_jL)=\left(\prod_{\gamma=1}^{M_1}\frac{\omega_\gamma^{(1)}-\lambda_j+iJ/2}{\omega_\gamma^{(1)}-\lambda_j-iJ/2}\right)\prod_{\gamma=1}^{M}\frac{\chi_\gamma-\lambda_j+iJ/2}{\chi_\gamma-\lambda_j-iJ/2};$$

$$\prod_{t=r\pm1}\prod_{\beta=1}^{M_t}\frac{\omega_\gamma^{(r)}-\omega_\beta^{(t)}+iJ/2}{\omega_\gamma^{(r)}-\omega_\beta^{(t)}-iJ/2}=-\prod_{\beta=1}^{M_r}\frac{\omega_\gamma^{(r)}-\omega_\beta^{(r)}+iJ}{\omega_\gamma^{(r)}-\omega_\beta^{(r)}-iJ},\quad r=1,2,\ldots,f-1,\quad M_f\equiv0,\quad \{\omega_\gamma^{(0)}\}=\{\lambda_j\}.$$

$$\frac{\chi_\gamma+1+iJS}{\chi_\gamma+1-iJS}\prod_{j=1}^{fN}\frac{\chi_\gamma-\lambda_j+iJ/2}{\chi_\gamma-\lambda_j-iJ/2}=-\prod_{\beta=1}^{M}\frac{\chi_\gamma-\chi_\beta+iJ}{\chi_\gamma-\chi_\beta-iJ}.$$

The ground state of the system is a flavor singlet described, in the limit $L\to\infty$, by the string solution[6]

$$\{\omega_\gamma^{(r)},\ \gamma=1,2,\ldots,M_r\}=\{p_A/\Lambda+iJ[(f-r+1)/2-g];\ q=1,2,\ldots,f-r,\ A=1,\ldots,N\},$$
$$r=0,1,\ldots,f-1.$$

One can see that other states with uniform distributions of strings of shorter length have excitation energies of order $J^2\Lambda$. It follows that the spin interaction between the impurity and the electrons (but not the full Hilbert space of the electrons themselves) is effectively described, in the limit $\Lambda\to\infty$, by the "fused" equations

$$\frac{\chi_\gamma+1+iJS}{\chi_\gamma+1-iJS}\left(\frac{\chi_\gamma+ifJ/2}{\chi_\gamma-ifJ/2}\right)^N=-\prod_{\beta=1}^{M}\frac{\chi_\gamma-\chi_\beta+iJ}{\chi_\gamma-\chi_\beta-iJ}, \tag{4}$$

where $N\sim\Lambda L$.

Corresponding to the "fusion" occurring in the equations, the flavor strings fuse the associated field operators into higher spin composites, as follows from the form of the wave function in the presence of stringlike pseudomomenta. It is[6]

$$F=\exp\{-\tfrac{1}{2}\Lambda J\sum_{j<l}|x_j-x_l|+ip(x_1+\ldots+x_f)\}\times[\ldots].$$

As $\Lambda\to\infty$, a local composite is formed. The role of flavor now becomes clear, since in its absence the f string are not allowed and the system relaxes into a different Hilbert space.

Fused equations similar to (4) occur, for example, in the study of higher-spin magnetic chains.[7] There, however, the higher-spin operators occur in the initial Hamiltonian and a rather artificial interaction is required to ensure integrability. In our case we find that the flavor interaction is so strong that it drives the operators to form higher-spin composites, dynamically.

We now proceed to the calculation of the impurity free energy by means of the standard methods based on the "string picture."[8] The result, for temperature T and magnetic field H, is

$$F_{(S,f)}^{\text{imp}}=(1/2\pi)\int d\xi\,\operatorname{sech}(\xi+\ln T/T_0)\ln[1+\eta_{2S}^{(f)}(\xi)], \tag{5}$$

where the function $\eta_{2S}^{(f)}$ is a member of a set $\{\eta_n^{(f)},\ n=0,1\ldots,\infty\}$ satisfying a system of coupled integral equations[9]

$$\ln\eta_n^{(f)} = -2\delta_{n,f}e^{\zeta} + G\ln(1+\eta_{n+1}^{(f)}) + G\ln(1+\eta_{n-1}^{(f)}), \tag{6}$$

with $\eta_0^{(f)} \equiv 0$ and

$$[n+1]\ln(1+\eta_n^{(f)}) - [n]\ln(1+\eta_{n+1}^{(f)}) \to 2H/T, \quad n\to\infty.$$

Here G and $[2n]$ are the usual integral operators given by the kernels $[2\pi\cosh\zeta]^{-1}$ and $n[\pi^2n^2+\zeta^2]^{-1}$, respectively.[10] In Eqs. (5) and (6) we have taken the scaling limit $N/L\to\infty$, $J\to 0$ while holding $T_0=(N/L)\exp(-\pi/J)$ fixed. The variable ζ is related to χ by $\pi\chi/J=\zeta+\ln TL/N$.

The solution of the equations is unique and is given by functions $\eta_n^{(f)}(\zeta)$ which are monotonically decreasing in ζ for all n and tending to finite limits $\eta_{n\pm}^{(f)}$ as $\zeta\to\pm\infty$. The limits are given by

$$\eta_{n-}^{(f)} = \sinh^2[(n+1)H/T]\sinh^{-2}(H/T)^{-1}, \quad n=0,1,2,\ldots, \tag{7}$$

and

$$\eta_{n+}^{(f)} = \sin^2[(n+1)\pi/(f+2)]\sin^{-2}[\pi(f+2)] - 1, \tag{8a}$$

for $n<f$, while for $n\geqslant f$

$$\eta_{n+}^{(f)} = \sinh^2[(n+1-f)H/T]\sinh^{-2}(H/T) - 1. \tag{8b}$$

Consider now the high-temperature properties of the model. They are determined, for a spin-S impurity, by the properties of $\eta_{2S}^{(f)}$ in the limit $\zeta\to-\infty$. Just as in the one-flavor case[10] this limit is approached with power corrections, leading to

$$F_{(S,f)}^{(\mathrm{imp})} \xrightarrow[T\to+\infty]{} -T\ln\frac{\sinh(2S+1)H/T}{\sinh H/T} + \frac{B_1}{\ln T/T_0} + \cdots.$$

This is the weak-coupling regime, governed by the trivial fixed point at $J=0$. The free energy is that of an isolated spin S up to logarithmic corrections characteristic of asymptotic freedom. The nature of this point is unaffected by the presence of the flavor degrees of freedom.

On the other hand, flavor affects significantly the low-temperature properties of the model. These are determined by the behavior of $\eta_{2S}^{(f)}$ in the limit $\zeta\to+\infty$. As can be read off from Eqs. (8), the nature of the limit and of the approach to it now depends on the flavor degrees of freedom.

In the case $f<2S$, the limit $\eta_{2S}^{(f)}$ is again obtained with power corrections and we have

$$F^{\mathrm{imp}} \xrightarrow[T\to 0]{} -T\ln\frac{\sinh(2S+1-f)H/T}{\sinh H/T} + \frac{C_1}{\ln T/T_0} + \cdots.$$

This is the free energy of a spin $S'=S-\frac{1}{2}f$. In other words, the impurity spin is partially screened. The approach to the limiting value is logarithmic indicating a trivial fixed point at $J=\infty$. When $f=2S$ the screening is complete, and, as $T\to 0$ $F^{\mathrm{imp}}\sim T_0^{-1}(D_1T^2+SH^2)$. A_1, B_1, C_1, and D_1 are numerical constants.

A new asymptotic infrared behavior arises for $f>2S$. We begin to analyze it by considering the zero-temperature magnetization for small magnetic fields. In the zero-temperature limit the thermodynamic equations collapse into a single equation which describes the maximum spin excitations above the ground state, which consists of an f-string χ configuration. For small magnetic field H the impurity magnetization is given by

$$M^{\mathrm{imp}}(H\sim 0) = (\mu/2)\int_{-\infty}^{\ln H/T_0}dx\,\sigma_0^{\mathrm{imp}}(x), \tag{9}$$

where σ_0^{imp} is the impurity contribution to the ground-state density of f strings. Its Fourier transform $\tilde{\sigma}_0^{\mathrm{imp}}(p)$ is

$$\sinh(SJp)[2\cosh(Jp/2)\sinh(fJp/2)]^{-1}, \quad f\geqslant 2S$$

and

$$\exp[(f/2-S)J|p|][2\cosh(Jp/2)]^{-1}, \quad f\leqslant 2S.$$

In the limit $H\to 0$, the magnetization is dominated by the properties of $\tilde{\sigma}^{\mathrm{imp}}(p)$ at $p=0$. While for $f<2S$, $\tilde{\sigma}_0^{\mathrm{imp}}(p)$ is discontinuous at $p=0$ leading to $M_{(H\sim 0)}^{\mathrm{imp}} = 2\mu(S-\frac{1}{2}f)+O(\ln H/T_0)$, for $f\geqslant 2S$ the transform is analytic in p so that $M^{\mathrm{imp}}(H)$ is controlled by the pole at $p=-2i/f$ ($p=-i$ if $f=2S=1$). Hence

$$M^{\mathrm{imp}}(H) \sim \mathrm{const}\times\mu(H/T_0)^{2/f}, \quad H\to 0 \tag{10}$$

leading to the critical exponent $\delta=f/2$.

Consider now the equations of finite temperature. The functions $\eta_n^{(f)}$ approach their limit values exponentially, $\eta_n^{(f)}(\xi) \to \eta_{n+}^{(f)} + c_n e^{-\tau\xi}$, $\xi \to \infty$, $n<f$, with $0<\tau<1$. The coefficient τ is given by $\tau=1$ for $f=1$ and

$$\tau = (2/\pi)\arcsin[(\lambda_0/4)^{1/2}], \quad f>2, \tag{11}$$

where λ_0 is the smallest eigenvalue of the positive-definite eigenvalue problem for $b_n = c_n(1+\eta_{n+}{}^f)^{-1}$;

$$(2+1/\eta_{n+}{}^f)b_n - b_{n+1} - b_{n-1} = \lambda b_n, \quad n=1,2,\ldots,f-1 \tag{12}$$

with the boundary conditions $b_0 = b_f = 0$.

We can now deduce the low-temperature expression for the impurity free energy,

$$F_{(f,S)}{}^{\mathrm{imp}} = -\tfrac{1}{2}T\ln(1+\eta_{2S+1}{}^{(f)}) - \frac{b_{2S}}{2\cos(\pi\tau/2)}T(T/T_0)^{\tau} + \ldots \tag{13}$$

resulting in a specific heat characterized by a critical exponent $\alpha = -\tau$.

A similar calculation determines the exponent of the magnetic susceptibility γ.[11] It is found to be $\gamma = 1-\tau$.

The coefficient $\tau = \tau(f)$ depends only on the flavor and is given by $\tau(3) = \frac{4}{5}$, $\tau(4) = \frac{2}{3}$, $\tau(5) = 0.624\,515\,12$, $\tau(6) = \frac{1}{2}$, ... In the limit $f\to\infty$, $\tau \to (1+\sqrt{5})/2f$, since Eq. (12) is a discrete approximation to a Sturm-Liouville problem with the integrable potential $(\sin x)^{-2}$.

This calculation leads to a complete characterization of the critical properties of the nontrivial infrared fixed point.

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Note added.—We received a recent preprint[12] addressing the model from a different point of view. The degenerate Anderson model is studied in the limit $U \gg E_d - U(f-1)/2 \gg f\Gamma$, which produces the model for the case $f=2S$. The $T=0$ magnetization equation is obtained and compared to the equation of another effective higher-spin fermion model and thus solved. The results are then assumed to hold for arbitrary f and S at $T=0$.

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[11] We thank D. Kessler for carrying it out.

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