Possible Breakdown of the Alexander-Orbach Rule at Low Dimensionalities

Amnon Aharony and D. Stauffer[a]
Department of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel
(Received 26 January 1984)

Simple conditions are presented under which the fractal dimension of a random walk on an aggregate, d_w, is given by $d_w = D+1$, where D is the aggregate's fractal dimension. These conditions are argued (with one simple speculative assumption) to apply for $D < 2$, implying a breakdown of the Alexander-Orbach rule $d_w = 3D/2$. Existing results for percolation clusters, lattice animals, and diffusion-limited aggregates seem to favor our new rule.

PACS numbers: 64.60.Cn

Much recent interest has been centered on the *anomalous diffusion* expected on fractal structures.[1-5] On such self-similar structures, the anomalous diffusion coefficient scales with distance as $r^{-\theta}$, and thus the average distance traveled by a random walker ("the ant in the labyrinth") after t time steps is $r \propto t^{1/d_w}$, with $d_w = 2+\theta$.[1] The number of sites S visited by the walker then scales as $S \propto r^D$, where D is the fractal dimensionality of the aggregate, i.e., $S(t) \propto t^{\bar{d}/2}$, with $\bar{d} = 2D/d_w$.[3,4] Much of the recent activity in the field was stimulated by the empirical observation of Alexander and Orbach (AO)[3] that the *"superuniversal"* value $\bar{d} = \frac{4}{3}$ is rather accurate for the *infinite incipient cluster (IIC) at the percolation threshold* p_c, for spatial dimensionalities $2 \leqslant d$. Recently, Leyvraz and Stanley (LS)[5] presented plausibility arguments that the AO rule should hold exactly, at least for Cayley trees and thus at high dimensionalities.

At p_c, the fraction of sites on the IIC within a volume L^d (of linear scale L) is $L^{-\beta/\nu}$, and hence $D = d - \beta/\nu$.[6] The conductivity of the same volume scales as $L^{-\mu/\nu}$, and one has $\theta = (\mu - \beta)/\nu$.[1] Combined with the AO rule, these relations yield

$$\mu/\nu = \tfrac{3}{2}(d - \tfrac{4}{3} - \beta/3\nu). \tag{1}$$

Since β and μ cannot be negative, this relation clearly breaks down for $d < \frac{4}{3}$. Moreover, at $d = 1+\epsilon$ one expects[7] that $\mu/\nu = \epsilon$, in contradiction to Eq. (1). Therefore, there must exist a lower critical dimensionality, d_l, below which Eq. (1) breaks down (if it is correct at higher d). At $d=2$, Eq. (1) predicts that $\mu/\nu = 91/96 = 0.948$. Very recently, very accurate independent studies[8] yielded $\mu/\nu = 0.973 \pm 0.005$, 0.977 ± 0.010, and 0.977 ± 0.016, apparently excluding the AO value even at $d=2$. The present Letter presents an argument which complements the LS argument for low d, and suggests that the lower critical dimensionality d_l is about 2.1, where the fractal dimensionality $D = d - \beta/\nu$ is exactly equal to 2. Thus, the argument implies that the AO rule may not hold at $d=2$ (where $D = \frac{91}{48} < 2$). Explicitly, our argument suggests that for $D \leq 2$, and for sufficiently large systems, the AO rule $\bar{d} = \frac{4}{3}$ crosses over to

$$\bar{d} = 2D/d_w = 2D/(2+\theta) = 2D/(D+1), \tag{2}$$

i.e.,

$$d_w = D+1, \tag{3}$$

or $\theta = D-1$. For the IIC at p_c this implies that Eq. (2) is replaced (for $d < d_l \simeq 2.1$) by

$$\mu/\nu = d-1. \tag{4}$$

Equation (4) was earlier conjectured by Straley,[9] and even earlier by Levinshtein, Shur, and Efros.[10] However, these authors used different arguments, and predicted Eq. (4) for all d, which is clearly wrong near six dimensions (where $\mu/\nu = 6$ and not 5). Equation (4) is clearly true at $d = 1+\epsilon$,[7] and the values of μ/ν near 0.98 in Ref. 8 are probably due to a slow crossover from 0.948 [Eq. (1)] to 1 [Eq. (4)]. Although the simulations may test different stages of the crossover, its slow nature may still yield similar effective values.

Our rule (3) seems to be also preferred by other types of fractals: Wilke *et al.*[11] find $d_w = 2+\theta = 2.6 \pm 0.3$ for *lattice animals* in $d=2$. Our rule would predict $d_w = D+1 = 2.56$, while the AO rule yields $d_w = 2.34$. For animals, the borderline dimensionality $D=2$ occurs at $d_l = 3$, where Wilke *et al.* found $d_w = 3.4 \pm 0.4$, the apparent deviation from $D+1 = 3D/2 = 3$ probably being due to logarithmic corrections (expected at d_l). Meakin and Stanley[12] have recently looked on *diffusion-limited (Witten-Sanders) aggregates,* and found $\bar{d} = 1.2 \pm 0.1$, not far from our $\bar{d} = 2D/(D+1) = 1.26$. For $d=3$ their results agree better with the AO $\frac{4}{3}$.

The LS argument concentrates on the number of visited sites, $S(t)$, and is specific for the percolation IIC at p_c. LS create $S(t)$ simultaneously with growing the underlying IIC: At each time step, the ant may either move to one of the already visited sites, or try to move into one of G growth sites never investigated before. In the latter case, this site may either be unoccupied, whence G decreases by $\Delta G = -1$ and the number of blocked sites, B, increases by unity; or it is occupied, whence $\Delta G > 0$ and the ant moves there. If there are no correlations, then[5] G is dominated by random fluctuations: $G \propto (S+B)^{1/2} \propto S^{1/2}$. The probability to increase S per unit time is[4] $dS/dt \propto G/S$. Substitution of $G \propto S^{1/2}$ immediately yields $S \propto t^{2/3}$, independent of dimensionality d.

The above argument clearly breaks down at $d=1$: Then $S(t)$ simply counts the number of visited sites which form a straight segment, and hence $G(t) \equiv 2$, $dS/dt \propto 1/S$, and $S \propto t^{1/2}$ instead of $t^{2/3}$. The slower increase of $S(t)$ results from the fact that the ant spends a long time revisiting sites, and only rarely reaches (and then shifts) the boundary, where $\Delta G \equiv 0$. A possible generalization of this picture would assume that the set of visited sites is mainly concentrated within a D-dimensional hypersphere of radius R on the IIC, with a rather quick decay outside. Nearly all the points within the "hypersphere" have been visited, $S \propto R^D$, and the number of unvisited growth sites is now proportional to the number of cluster points on its *surface* (not its perimeter), $G \propto R^{D-1} \propto S^{1-1/D}$. In general, G might be contained in a layer of width ΔR. If the surface of $S(t)$ were rough, then one might have $\Delta R \propto R^x$, with some new exponent x.[13] Near $d=1$ our results show that $x=0$. Numerical pictures on diffusion-limited aggregates[12] at $d=2$ also seem to have $x=0$. We therefore *assume* no surface roughening, i.e., $x=0$, or $\Delta R = \text{const}$. This assumption will hopefully be checked in future numerical simulations. With this assumption $dS/dt \propto S^{-1/D}$, giving $S(t) \propto t^{D/(D+1)}$ and thus Eqs. (2) and (3). Comparing this result with $\theta = (\mu-\beta)/\nu$ and $D = d - \beta/\nu$ we get Eq. (4).

Next we present a plausibility argument that the result (2) should[14] hold for $t \gg 1$ and $D < 2$. At short times, the growth of $S(t)$ may be described by the random changes in G, as given by LS. However, at low dimensionalities, there are many loops on all length scales, and every site is visited many times. Therefore, it is reasonable to conclude that the set of visited sites gradually becomes *compact,* covering most of the sites (on the infinite cluster) within a linear scale $R \propto S^{1/D}$. (Random percolation clusters below and at the threshold are not compact,[15] but the set of visited sites is not random. Instead it might be similar to the "growing animals" of the Eden process[15] which are compact.) The number of possible growth sites can therefore not exceed the number of surface sites $S^{1-1/D}$. For $D < 2$ this limit is smaller than the statistical $S^{1/2}$, and thus G will be dominated by the surface term: $G \propto S^{(D-1)/D}$. For $D > 2$, on the other hand, the statistical term wins: $G \propto S^{1/2}$ as in the LS theory.

Where, for $d=2$, should the crossover between the two rules occur as a function of t? Since then $S^{1/2}$ and $S^{1-1/D}$ differ by only a factor $S^{1/28} = S^{5/182}$, proportional to $t^{10/243}$ according to the AO rule, only at extremely long times, i.e., for vary large S, does the "ant" notice the deviation from the LS theory. Random percolation clusters at the threshold of the square lattice[15] have a diameter near $S^{1/D}$ which is 350 (the maximum width simulated by Zabolitzky) for S near 66 550, corresponding to $t = 23$ million steps; but still the ratio $S^{5/182}$ is then less than 1.4. Thus one may start to see deviations from the LS and AO rules, as observed in Ref. 8, but one does not yet see the asymptotic behavior for $S^{5/182} \gg 1$. Only the beginning of the crossover was felt so far. Note, however, that the Monte Carlo simulations of Ref. 14 for the two-dimensional number of visited sites gave $S \propto t^{0.65+0.01}$, in good agreement with our asymptotic prediction $t^{D/(D+1)} = t^{91/139} = t^{0.655}$ though still compatible with $t^{2/3}$.

In conclusion, we emphasize again that we have not presented a rigorous proof of Eq. (3), nor shown without doubt that the AO rule is wrong. In the same spirit as LS, we presented plausibility arguments favoring Eqs. (3) and (4) for $D < 2$. In the spirit of AO, we propose Eq. (3) as an empirical rule, to be checked by further analytical and numerical studies.

We thank the authors of Refs. 5, 8, and 12 for advance information on their work, and A. Coniglio, F. Leyvraz, and H. E. Stanley for helpful comments on the manuscript. This paper was supported by a grant from the U.S.–Israel Binational Science Foundation (BSF). One of us (D.S.) acknowledges support from the Deutsche Forschungsgemeinschaft for his visit at Tel Aviv University.

Note added.—After we submitted this paper we found out that Eq. (2) was also derived by Alexander.[16] However, the argument for the lower critical dimensionality $D=2$ appears here for the first time.

---

[a]Present and permanent address: Institut fur Theoretische Physik, Universität Köln, D-5000 Köln 41, West

Germany.

[1]Y. Gefen, A. Aharony, and S. Alexander, Phys. Rev. Lett. **50**, 77 (1983).

[2]D. Ben-Avraham and S. Havlin, J. Phys. A **15**, L691 (1982).

[3]S. Alexander and R. Orbach, J. Phys. (Paris), Lett. **43**, L625 (1982).

[4]R. Rammal and G. Toulouse, J. Phys. (Paris), Lett. **44**, L13 (1983).

[5]F. Leyvraz and H. E. Stanley, Phys. Rev. Lett. **51**, 2048 (1983).

[6]E.g., A. Kapitulnik, A. Aharony, G. Deutscher, and D. Stauffer, J. Phys. A **16**, L269 (1983).

[7]S. Kirkpatrick, Phys. Rev. B **15**, 1533 (1977).

[8]J. G. Zabolitzky, Phys. Rev. B (to be published). See also C. J. Lobb and D. J. Frank, Phys. Rev. B (to be published); H. J. Herrmann, B. Derrida, and J. Vannimenus, Phys. Rev. B (to be published); D. C. Hong, S. Havlin, H. J. Herrmann, and H. E. Stanley, Phys. Rev. B (to be published).

[9]J. Straley, J. Phys. A **13**, 819 (1980).

[10]M. E. Levinshtein, M. S. Shur, and E. L. Efros, Zh. Eksp. Teor. Fiz. **69**, 386 (1976) [Sov. Phys. JETP **42**, 197 (1976)].

[11]S. Wilke, Y. Gefen, V. Ilkovic, A. Aharony, and D. Stauffer, J. Phys. A **17**, 647 (1984). See also S. Havlin, Z. Djordjevic, I. Majid, H. E. Stanley, and G. H. Weiss, to be published.

[12]P. M. Meakin and H. E. Stanley, Phys. Rev. Lett. **51**, 1457 (1983).

[13]A. Coniglio and H. E. Stanley, Phys. Rev. Lett. **52**, 1068 (1984).

[14]J. C. Angles d'Auriac, A. Benoit, and R. Rammel, J. Phys. A **16**, 4039 (1983).

[15]H. P. Peters, D. Stauffer, H. P. Holters, and K. Lowenich, Z. Phys. B **34**, 399 (1979), ~~and references~~ therein; M. Plischke and Z. Rácz, to be published.

[16]S. Alexander, Ann. Israel Phys. Soc. **5**, 149 (1983).