Richardson Number Criterion for the Nonlinear Stability of Three-Dimensional Stratified Flow

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With use of a method of Arnol'd, we derive the necessary and sufficient conditions for the formal stability of a parallel shear flow in a three-dimensional stratified fluid. When the local Richardson number defined with respect to density variations is everywhere greater than unity, the equilibrium is formally stable under nonlinear pertrubations. The essential physical content of the nonlinear stability result is that the total energy acts as a "potential well" for deformations of the fluid across constant density surfaces; this well is required to have definite curvature to assure stability under these deformations.

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With use of a method of Arnol'd[1] and others,[2,3] we have investigated the nonlinear stability of two- and three-dimensional incompressible flows of an inviscid stratified fluid treated as a Hamiltonian system. In this note, we report on the application of this technique to the important case of a shear flow with velocity profile $U(z)$, and density profile $\rho(z)$. We do not present the full set of conditions for nonlinear stability of this flow, but do exhibit the necessary and sufficient conditions for the formal stability of the flow. Formal stability means that a certain functional of the flow fields is definite in sign. Given formal stability, nonlinear stability requires additional convexity estimates to be satisfied. These do not alter the physical implications of the conditions derived here.[3]

The two-dimensional analysis[4] of the stratified fluid equations linearized about a planar shear flow $U(z)$, $\rho(z)$, shows that neutral stability (purely imaginary spectrum) occurs provided the Richardson number is everywhere greater than $\frac{1}{4}$. Here we derive the analogous criterion for formal stability for three-dimensional nonlinear deformations of the flow. Our criterion is that the local Richardson number defined with respect to variations across constant-density surfaces must be greater than 1. This focuses attention on the realm between $\frac{1}{4}$ and 1 for intensive theoretical and experimental investigation.

We treat stability in the Boussinesq approximation[4] for incompressible flow. See Ref. 2 for the treatment of nonlinear stability for compressible flows, and Ref. 3. for incompressible, stratified, non-Boussinesq flows. We address solutions of the momentum equation

$$\frac{\partial}{\partial t}\vec{u}+(\vec{u}\cdot\nabla)\vec{u}=-\nabla p-\rho g\hat{z}, \tag{1}$$

along with

$$\frac{\partial}{\partial t}\rho+\vec{u}\cdot\nabla\rho=0 \quad\text{and}\quad \nabla\cdot\vec{u}=0, \tag{2}$$

in a domain on whose boundary the normal component of the velocity $\vec{u}$ must vanish and the density ρ must be constant. In (1) and (2), p is the pressure and g is the constant gravitational acceleration in the $-\hat{z}$ direction. The constant reference density multiplying the acceleration in (1) has been set equal to unity.

Solutions to these equations conserve the energy

$$\int d^3x\,[\tfrac{1}{2}|\vec{u}|^2+\rho g z]. \tag{3}$$

Both ρ and the potential vorticity

$$q=(\nabla\times\vec{u})\cdot\nabla\rho \tag{4}$$

are conserved along fluid particle trajectories. Thus, for an arbitrary function $G(q,\rho)$,

$$A(\vec{u},\rho)=\int d^3x\,[\tfrac{1}{2}|\vec{u}|^2+\rho g z+G(q,\rho)+\lambda q] \tag{5}$$

is conserved. The term λq in (5) is separated to cancel some boundary terms which arise below. The role of the function $G(q,\rho)$ is that of a familiar Lagrange multiplier expressing the constraints on the flow imposed by conservation of q and ρ.

We now examine the first variation of $A(\vec{u},\rho)$ and relate its critical points to stationary solutions $\vec{u}_e$. The first variation is

$$\delta A(\vec{u}_e,\rho_e)=\int d^3x\,\{\delta\vec{u}\cdot[\vec{u}_e-G_{qq}\nabla\rho_e\times\nabla q_e]+\delta\rho[gz+G_\rho(\nabla\times\vec{u}_e)\cdot\nabla G_q]\}+(\lambda+G_q)|_S\int ds\hat{n}\cdot\{\delta\rho\,\nabla\times\vec{u}_e-\nabla\rho_e\times\nabla\vec{u}_e\}, \tag{6}$$

where $G_\rho=\partial G/\partial\rho$ evaluated at $q_e\rho_e$, etc., S is the boundary surface of the domain of the flow, and $\hat{n}$ is the outward unit normal vector on S.

δA in (6) vanishes at $\vec{u}_e$, ρ_e satisfying

$$\vec{u}_e=G_{qq}\nabla\rho_e\times\nabla q_e, \tag{7}$$

$$gz+G_\rho=(\nabla\times\vec{u}_e)\cdot\nabla G_q \tag{8}$$

in the interior, and

$$\lambda=-G_q \tag{9}$$

on the boundary. Flows satisfying (7) and (8) can be verified to be stationary solutions of (1) and (2). Expression (7) implies the requirements $\vec{u}_e\cdot\nabla\rho_e=\vec{u}_e\cdot\nabla q_e=0$ for stationary flows; (8) is the three-dimensional analog of Long's equation.[5]

We use (7) and (8) to determine $G(q_e,\rho_e)$ in terms of the Bernoulli function

$$K(q_e,\rho_e)=p_e+\rho_e g z+\tfrac{1}{2}|\vec{u}_e|^2, \tag{10}$$

via

$$G(q_e,\rho_e)=q_e\int^{q_e}\frac{dx}{x^2}K(x,\rho_e)+q_e\gamma(\rho_e), \tag{11}$$

where $\gamma(\rho_e)$ is an arbitrary function of ρ_e.

An equilibrium flow is said to be *formally stable* if the second variation of $A(\vec{u},\rho)$ at the critical point $\vec{u}_e,\rho_e$ is definite in sign. Formal stability implies[3] linearized stability since definiteness of δ^2A gives a preserved norm for the linearized solutions. As noted, nonlinear stability requires both formal stability and some convexity conditions on the function $G(q,\rho)$. For the present case, we find

$$\delta^2A(\vec{u}_e,\rho_e)=\int d^3x\left\{|\delta\vec{u}|^2+(\delta q,\delta\rho)\begin{bmatrix}G_{qq} & G_{q\rho}\\ G_{q\rho} & G_{\rho\rho}\end{bmatrix}\begin{bmatrix}\delta q\\ \delta\rho\end{bmatrix}\right\}. \tag{12}$$

From this we see that a *sufficient* condition for formal stability is that the eigenvalues of the two-by-two matrix in (12) are positive; namely,

$$G_{qq}>0, \tag{13}$$

and

$$G_{qq}G_{\rho\rho}-G_{q\rho}{}^2>0. \tag{14}$$

We can sharpen these sufficient conditions, however, by noting that $\mathrm{div}\nabla\cdot\delta\vec{u}=0$, so there are only two independent components of $\delta\vec{u}$, which along with $\delta\rho$ allow us to cast the definiteness of δ^2A into a linear three-by-three operator eigenvalue condition, whose eigenvalues must then be either all positive or all negative. This condition is made explicit in the example we now discuss.

Our example is the parallel equilibrium flow

$$\vec{u}_e(\vec{x})=(u(y,z),0,0), \tag{15}$$

$$\rho_e(\vec{x})=\rho(z). \tag{16}$$

This is a standard configuration and application of the Arnol'd method to it provides insight into the value of the technique. The validity of the linearized results on this flow have been examined in laboratory and geophysical situations. Our nonlinear result will thus provide impetus for further experimental study of these important flows. We separate the y and z dependences in $u(y,z)$ into a small, slowly varying y dependence plus a general z dependence $U(z)$. Thus, we write

$$u(y,z)=f(y)+U(z). \tag{17}$$

The role of $f(y)$ is to break the $q_e=0$ degeneracy of the two-dimensional $f=0$ flow, which is the conventional setup. The physical situation we wish to describe is a shear flow $U(z)$ with a smooth, small $f(y)$ imposed upon it to give the three-dimensionality needed for $q_e \neq 0$. We wish to parametrize $f(y)$ by a velocity scale, f_0, which is much less than $U(z)$, and by a length scale L which is large compared to any other lengths in the problem. We choose

$$f(y) = f_0(y/L)^2; \quad f_0 << U(z); \tag{18}$$

and restrict the domain of y to be $|y| << L$. In what follows, we expand all quantities in L^{-1}, capturing the essence of the stability problem in the leading orders of L which are retained for L very large.

From the Bernoulli function, (10), we find (dropping the subscript e henceforth)

$$G(q,\rho) = -[p + \rho g z + \tfrac{1}{2}U^2(z)] + \tfrac{1}{2}G_{qq}q^2 + O(q^4), \tag{19}$$

with

$$G_{qq} = \frac{u}{u_{yy}\rho_z^2} = \frac{L^2 U(z)}{f_0 \rho_z^2}\left[1 + \frac{f(y)}{U(z)}\right], \tag{20}$$

and we drop the last term commensurate with our assumptions on $f(y)$. G_{qq} is now a function of z (or ρ) alone. q in our flow is

$$q = (f_0/L)(y/L)(-\rho_z). \tag{21}$$

Since q is small for $|y| << L$, the neglect of higher-order terms in q, wherever they occur, is an excellent approximation.

Now we choose the two independent components of $\delta\vec{u}$ in (12) from the vertical velocity $v_3(\vec{x},t) = \delta\vec{u}\cdot\hat{z}$ and the vorticity $\omega_3(\vec{x},t) = (\nabla \times \delta\vec{u})\cdot\hat{z}$. This choice is motivated by the observation that the only essential dependence on the equilibrium flow is on the vertical coordinate z.[6] To leading order in L^{-1} a calculation shows that $\delta^2 A(\vec{u}_e, \rho_e)$ is given by

$$\delta^2 A(\vec{u}_e,\rho_e) = \int d^3x\,(v_3,\omega_3,\delta\rho)\begin{bmatrix} \nabla^2/\nabla_\perp^2 & 0 & 0 \\ 0 & -\dfrac{1}{\nabla_\perp^2} + \rho_z^2 G_{qq} & \rho_z U_z G_{qq}\partial_y \\ 0 & -\rho_z U_z G_{qq}\partial_y & G_{\rho\rho} - U_z^2 G_{qq}\partial_y^2 \end{bmatrix}\begin{bmatrix} v_3 \\ \omega_3 \\ p\delta_\rho \end{bmatrix}, \tag{22}$$

with $\nabla_\perp^2 = \partial_x^2 + \partial_y^2$ and $\nabla^2 = \nabla_\perp^2 + \partial z^2$. Precise meaning to $(\nabla_\perp^2)^{-1}$ is given by imposing periodic boundary conditions in x and y for each of v_3, ω_3, and $\delta\rho$. A term $f_y \partial_z \delta\rho$ has been neglected relative to $U_z \partial_y \delta\rho$, which is retained. This ordering means our choice of L must be large enough to overcome any very large vertical wave numbers in $\delta\rho$. The arbitrary function $\gamma(\rho_e)$ in (11) is set to zero.

For formal stability, we demand that $\delta^2 A$ be of definite sign for all independent variations in $(v_3, \omega_3, \delta\rho)$ space. That sign must be positive, as we see by looking in the direction $(v_3, 0, 0)$. Then by looking in the direction $(0, \omega_3, \delta\rho)$ we learn that the necessary and sufficient conditions for formal stability are that the two-by-two submatrix operator in (22) have only positive eigenvalues. This requirement is most easily expressed by Fourier transforming in x and y to wave numbers k_1 and k_2. The two-by-two submatrix becomes algebraic, and positivity of its eigenvalues occurs if and only if

$$1/k_\perp^2 + \rho_z^2 G_{qq} > 0, \tag{23}$$

and

$$G_{\rho\rho}[1 + k_\perp^2 \rho_z^2 G_{qq}] + k_2^2 U_z^2 G_{qq} > 0, \tag{24}$$

with $k_\perp^2 = k_1^2 + k_2^2$.

Since we allow arbitrary variations of v_3, ω_3, and $\delta\rho$, each of k_1 and k_2 can be as large as we like. This means that we must have

$$\rho_z^2 G_{qq} = u/u_{yy} > 0, \tag{25}$$

and

$$G_{\rho\rho} > \max_{(k_1,k_2)}\left[\frac{k_2^2 U_z^2 G_{qq}}{1 + k_\perp^2 \rho_z^2 G_{qq}}\right] = 0. \tag{26}$$

The first of these is the usual Rayleigh criterion for stability of shear flows in y. Its presence here is expected since we have no stratification in the horizontal direction. Condition (26) is the desired Richardson-number criterion. Note that

$$G_{\rho\rho} = -g\,\partial z/\partial\rho - \partial^2[\tfrac{1}{2}U^2(z)]/\partial\rho^2. \tag{27}$$

When $(U^2)_{\rho\rho}$ is positive, we may define the generalization of the usual Richardson number to be

$$N_{\mathrm{Ri}}(z) = N(z)^2/\{\rho_z^2 \partial^2[\tfrac{1}{2}U^2(z)]/\partial\rho^2\}, \tag{28}$$

with $N^2(z) = -g\,\partial\rho/\partial z$ the Brünt-Väisäla frequency in Boussinesq approximation. [N_{Ri} defined by (28) agrees locally with the standard gradient definition, if one uses the linearization of U and ρ (e.g., Ref. 3)]. The necessary and sufficient condition for formal stability then becomes

$$N_{\mathrm{Ri}}(z) > 1 \tag{29}$$

everwhere in the flow. This is our central result.

In addition, there are situations where ρ_z positive (a statically unstable configuration) may be stabilized by the shear flow. To exhibit this stabilization, we assume $\rho_z \neq 0$ and define the "inverse Richardson number"

$$a(z) = \{\partial^2[\tfrac{1}{2}U^2(z)]/\partial\rho^2\}(-\rho z/g). \tag{30}$$

When $\rho_z < 0$, that is for statically stable stratification, all flows with $a(z) < 1$ are formally stable. When $\rho_z > 0$, that is for *statically unstable stratification,* all flows with $a(z) > 1$ are formally stable. The first case is usually understood by saying that the kinetic energy acquired by a parcel of fluid crossing density surfaces is not sufficient to overcome the potential energy required to move the parcel. The second case is less familiar and is only possible if second derivatives of U are relatively large. In this case, the potential energy that would be gained by a fluid parcel in crossing density surfaces is not sufficient to overcome kinetic energy lost in the same traverse.

The essence of our argument in this note is that the negative of the Bernoulli function (10) acts as a "potential well" for stratified flow. This is seen in (19) where G is, for this heuristic discussion, $-(p+\rho gz+\frac{1}{2}|\vec{u}|^2)$. Our requirement that $G_{\rho\rho} > 0$ tells us that this potential well has positive curvature for crossing density surfaces, when the flow is formally stable. This note provides detailed demonstration of this notion, which itself was discussed as long ago as 1931 by Prandtl.[7]

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[1]V. I. Arnol'd, Dokl. Nat. Nauk. **162**, 773 (1965), and Am. Math. Soc. Transl. **79**, 267 (1969).

[2]D. D. Holm, J. E. Marsden, T. Ratiu, and A. Weinstein, Phys. Lett. **98A**, 15 (1983), and to be published.

[3]H. D. I. Abarbanel, D. D. Holm, J. E. Marsden, and T. Ratiu, "Nonlinear Stability Analysis of Ideal Stratified, Incompressible Fluid Flow" (to be published).

[4]P. G. Drazin and W. H. Reid, *Hydrodynamic Stability* (Cambridge Univ. Press, Cambridge, England, 1981). Especially important for us is the work of Miles and Howard reported in Sect. 44.

[5]R. R. Long, Tellus **5**, 42 (1953). See also the monograph by C. S. Yih, *Stratified Flows* (Academic, New York, 1980).

[6]See, for example, the discussion of thermal convection in S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Cambridge Univ. Press, Cambridge, England, 1961).

[7]L. Prandtl, *Führer durch die Strömungslehre* (Vieweg, Braunschweig, 1931), Sec. V, 12(d).