SU(3) Heavy-Quark Potential with High Statistics

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The results of a high-statistics calculation of the SU(3) heavy-quark potential are presented. The validity of scaling is tested quantitatively. New results are given for the string tension and the strength of the long-range Coulomb term in the potential.

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We have recently completed a series of high-statistics Monte Carlo evaluations of large, rectangular Wilson loops in SU(3) lattice gauge theory. By use of the standard one-plaquette action, all planar loops up to 6×9 in size were measured[1] on a $12^3\times 16$ lattice, at a set of β values ($\beta = 6/g^2$) ranging from 5.8 to 7.6. A full discussion of the results will be presented elsewhere. In this paper, we address the large distance, nonperturbative physics in the heavy-quark potential, and give quantitative results on the validity of scaling.[2]

In the continuum limit, the heavy-quark potential can be written

$$V(R) = s(R/\xi)/R, \tag{1}$$

where s is a dimensionless scaling function, and

$$\xi = \frac{c}{\Lambda_0} = c\alpha\left(\frac{8\pi^2\beta}{33}\right)^{-51/121}\exp\left(\frac{4\pi^2\beta}{33}\right). \tag{2}$$

For the quark potential the string tension, K, sets the physical length scale, and it is natural to adjust the constant c so that $\xi\sqrt{K}\sim 1$. Formulas (1) and (2) become exact in the asymptotic limit, $\xi/\alpha\to\infty$ (or $\beta\to\infty$). It is clearly a vital question to know how fine grained the lattice must be, or equivalently how large a β is needed, before (1) and (2) become good approximations. In our work, the answer is $\beta\geqslant 6.0$. Our smallest β values are $\beta = 5.8$, 6.0, 6.1, and 6.2. The points with $\beta\geqslant 6.0$ are consistent with (1) and (2), but inclusion of $\beta = 5.8$ leads to clear violations of scaling. At $\beta = 6.0$, $\xi\sim 4\alpha$ (see the discussion of K below), thus $\alpha\sim 1/4\sqrt{K}$ is the coarsest lattice we can tolerate. Once in the scaling region, we must also control finite volume effects, which depend on ξ/L, where L is the shortest lattice dimension. From $\xi\sim 4\alpha$ at $\beta = 6.0$, we have, assuming scaling, $\xi\sim 6.5\alpha$ at $\beta = 6.4$, and so to keep L approximately twice ξ or greater, we imposed a maximum of 6.4 on β. The choice of a safety factor of 2 between L and ξ is somewhat arbitrary. A more specific requirement is that we want to avoid crossing the deconfining transition. We arrived at a similar upper bound on β by roughly estimating the location of this transition. Consider first the finite temperature configuration, $L^3\times T$, $T << L$. From a recent calculation,[3] we have that the lattice is in the confined phase for $T \gtrsim (66\Lambda_0)^{-1}$. If we make the plausible assumption[4] that for an $L^3\times T$ lattice with $T > L$, the shorter dimension L should now play the role of inverse "temperature," then to be in the confined phase we estimate $L\gtrsim (66\Lambda_0)^{-1}$, which translates into $\beta\lesssim 6.55$ for our $12^3\times 16$ lattice.

For the interquark separation, R, the formal continuum limit is $R >> \alpha$, and explicit calculations must be done to see if continuum behavior can be found accurately at finite R/α. There is good evidence from the study of correlation functions in spin systems that once the coupling is in the scaling region, the asymptotic formulas valid for $R >> \alpha$ also become very accurate for finite $R/\alpha > 1$.[5] In our work, we can measure the heavy-quark potential for $R/\alpha = 1$–6. Consistent with the spin system results, we find that continuum behavior is reached rapidly as R/α increases, and we need eliminate only the smallest value, $R/\alpha = 1$ (we have checked that the χ^2 of our fits to the potential increases unacceptably if $R/\alpha = 1$ is included). To summarize, we have imposed two conditions, $6.0\leqslant\beta\leqslant 6.4$ and $2\leqslant R/\alpha\leqslant 6$. Meeting them hopefully allows a reliable calculation on our $12^3\times 16$ lattice, in which the worst effects of the finite lattice grid and volume have been eliminated.

As discussed above, the correlation length, ξ, changes only by a factor of approximately 1.5 between $\beta = 6.0$ and 6.4. This means that in order to set quantitative limits on scaling, we need to gather very accurate data for Wilson loops. This was partially accomplished by making long runs. At each β value ($\beta = 6.0$, 6.1, 6.2, and 6.4), the lattice was swept at least 700 times. The link update algorithm was a mild variation on that of Cabbibo and Marinari,[6] in which all three SU(2) subgroups of SU(3) were used. At least 200 sweeps were dropped from the beginning of each run to insure equilibrium. Much more important than the length of the runs in reducing statistical errors was the way in which the Wilson loop data were gathered. We used the variance reduction method applied recently by Parisi, Petronzio, and Rapuano[7] in a calculation of thermal Wilson loops. Described in purely operational terms, we proceeded as follows. Measurements of Wilson loops were made every ten sweeps of the lattice. In the measurement of $W(R,T)$ for $R \geqslant 2\alpha$, the links in the T direction of the loop were integrated out.[1] In practice, this was done by updating these links an additional fifteen times, holding the surrounding links fixed. The value of the loop was then calculated in the usual manner, except that for the T direction, each link matrix, U, was replaced by its average over the fifteen additional updates. As discussed in Ref. 7, what is going on is the replacement of a fluctuating observable by another with the same expected value, but with smaller variance. Using this method, we observed a reduction in statistical error by a factor of approximately 10 for a typical large loop like 4×8, giving an effective gain in computational power of 100. Similar variance reduction methods have recently been applied for the SU(2) case.[8]

From the Wilson loop values, we proceed to construct the potential in a manner similar to previous calculations.[9,10] First, lattice potentials, $V_l(R)$, were determined at each β by fitting $\ln[W(R,T)]$ to a straight line in T for $T > R$. As seen in Fig. 1, there is very good evidence that $W(T,R)$ behaves

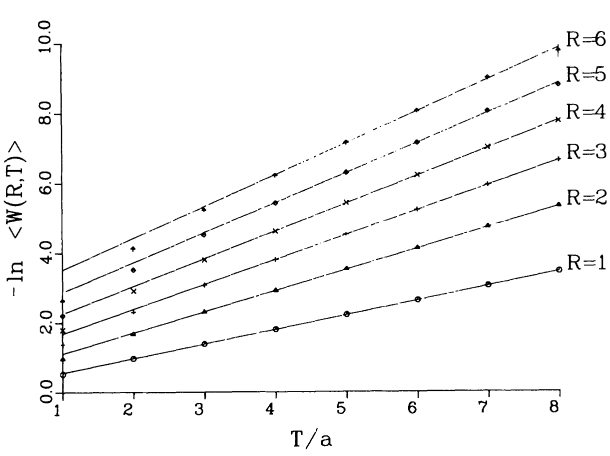

FIG. 1. $-\ln\langle W(R,T)\rangle$ vs T/α for $\beta = 6.0$.

exponentially in T for $T > R$, which means that the heavy-quark-antiquark system is in its ground state. This is our main reason for favoring rectangular loops over those which wrap around the lattice in determining the potential. With rectangular loops, the exponential behavior in T is explicitly verified, and then several values of T feed into the determination of $V_l(R)$ at a given R value. This allows much more accurate results to be obtained.

With our results for the lattice potentials in hand, we are ready to address the question of scaling. In the continuum limit, any physical quantity with dimensions of mass becomes a multiple of Λ_0. The dominant dependence on β of any such quantity resides in the exponential factor of Eq. (2). To parametrize possible deviations from continuum scaling, we define a family of Λ_0 parameters by

$$\Lambda_0(f) = \frac{1}{\alpha}\left(\frac{8\pi^2\beta}{33}\right)^{51/121} \exp\left(-\frac{4\pi^2\beta f}{33}\right), \quad (3)$$

and our goal will be to see what limits can be set on the parameter f. The power behavior in β multiplying the exponential in (3) is purely decorative at this point, and plays a negligible numerical role in our analysis of scaling. We carry it along simply because for $f = 1.0$, $\Lambda_0(f)$ reduces to the universally used formula, and our results can be compared easily with other calculations.

The potentials $V_l(R)$ contain an explicitly α-dependent self-energy, V_0, which must be removed before the continuum potential, V, can be found. This was done by the procedure described in Refs. 9 and 10, to which the reader is referred for details. Here, we just mention that in the step involving the force, the points $R = 2\alpha$ and 3α were used. The correlation length was defined by $\xi = c/\Lambda_0(f)$. The constant c is arbitrary, but to maintain $\xi\sqrt{K} \sim 1$, we chose $c + 0.011$, which was our previous value for $\Lambda_0/\sqrt{K}$.[10] The self-energy subtraction procedure produces a set of ξV values for any ξ; all that is needed is a definite value for ξ/α at each value of β. If an incorrect dependence of ξ/α on β has been assumed, the plot of ξV vs $x = R/\xi$ will show scatter and fail to map out a smooth curve. This can be seen by eye if a value of f far from 1.0 is used, but for quantitative results one must fit to a specific analytic form for ξV to define "smoothness." Since we have taken care to stay in the confined region, and have eliminated the shortest distance points, a simple linear-plus-Coulomb form is justified, where the Coulomb term is associated with long-range nonpertrubative physics, not gluon exchange. For each value of the parameter f, we performed a least-squares fit of ξV to the three-parameter form

$$\xi V = -(\alpha/x) + Ax + B. \tag{4}$$

We then searched in f to find the value, f_0, which gave an overall minimum in χ^2. The uncertainty in f_0 was determined by using the criterion[11] that $f_0 \rightarrow f_0 \pm \delta f$ should cause a unit increase in χ^2. The result is that $f_0 = 1.05 \pm 0.08$. The total χ^2 at $f_0 = 1.05$ is 13.8, which is quite acceptable for a fit with 17 degrees of freedom. Note that the perfect continuum scaling value of $f = 1.0$ is also totally acceptable statistically. Large deviations from $f = 1.0$ cause χ^2 to increase to very improbable values. With 90% confidence, values of f greater than 1.32 or less than 0.89 are ruled out.

The values of α and A found in the fits vary only slightly as f varies over the range $f_0 \pm \delta f$, and we quote here only the $f = 1.0$ results. At $f = 1.0$, we get $\alpha = 0.25 \pm 0.01$ and $A = 1.38 \pm 0.04$, which translates, with $K = A/\xi^2$, into $\Lambda_0/\sqrt{K} = (9.4 \pm 0.1) \times 10^{-3}$. The errors quoted so far are purely statistical, coming from the fit to the data with $6.0 \leq \beta \leq 6.4$ and $2 \leq R/\alpha \leq 6$. There is additional uncertainty in α and $\Lambda_0/\sqrt{K}$, caused by some sensitivity to the exact choice of points in the fits. For example, one can be ultraconservative and eliminate $\beta = 6.4$, or more daring and include $\beta = 6.8$, etc. We attempt to take account of this in our best estimates of the final results for $f = 1.0$:

$$\alpha = 0.25 \pm 0.02, \qquad \Lambda_0/\sqrt{K} = (9.4 \pm 0.3) \times 10^{-3}. \tag{5}$$

In Fig. 2, we show our results for $V/\sqrt{K}$ vs $K^{1/2}R$ for the present calculation, along with the linear-plus-Coulomb fit. In Fig. 3, we show all of our data for $\beta = 6.0$. Note that points with $\beta \geq 6.4$ continue to map out a smooth curve. In contrast, if plotted, the $\beta = 5.8$ points with $R/\alpha > 3$ would be high by several standard deviations. Thus the distortion of scaling due to the lattice spacing growing too large seems more violent than effects associated with finite lattice volume.

Our value of α is within estimated errors of $\pi/12$, the value expected from transverse flux tube vibrations.[12,13] Although we are working at $K^{1/2}R \sim 1$, rather than $K^{1/2}R >> 1$, we find this highly suggestive. We plan to map out the chromoelectric flux distribution represented by our potential in future work.

With regard to the string tension, our result (5) for $\Lambda_0/\sqrt{K}$ is quite close to that obtained in another recent high-statistics calculation,[14] which gave $\Lambda_0/\sqrt{K} = 9.6 \times 10^{-3}$. It is also consistent with our previous calculation[10] which, on an $8^3 \times 12$ lattice, gave $\Lambda_0/\sqrt{K} = (11 \pm 3) \times 10^{-3}$. At $\beta = 6.0$, (5) gives $1\sqrt{K} = 4.01\alpha$. This was the basis of our earlier remark that $\xi \sim 4\alpha$ at $\beta = 6.0$.

We can draw two important conclusions from this work. First, with sufficiently accurate data, quantitative tests can be carried out on the validity of scaling. Second, for the case of SU(3) with the one-plaquette action, there is good evidence for scaling according to (1) and (2), once $\xi/\alpha \gtrsim 4$. The slight

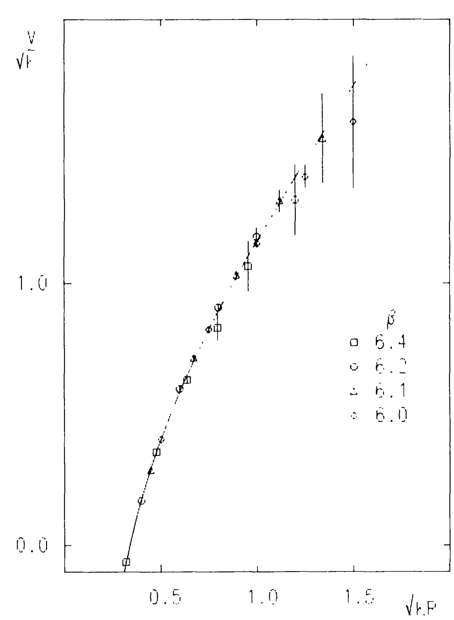

FIG. 2. Linear-plus-Coulomb fit to the data with $6.0 \leq \beta \leq 6.4$.

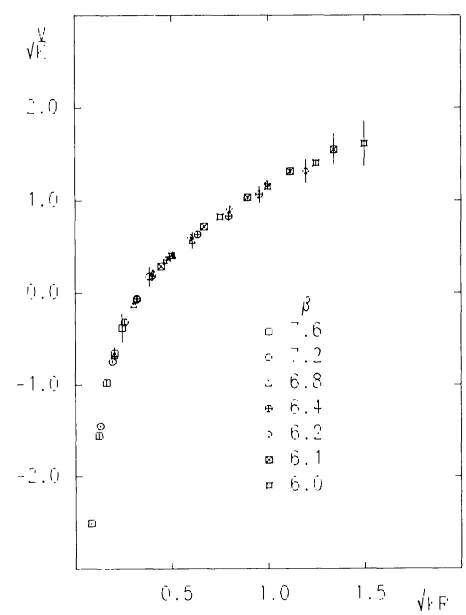

FIG. 3. The quark potential for all the data with $\beta \geq 6.0$.

deviation from $f = 1.0$ found in our best fit was caused mainly by the data at $\beta = 6.4$. We are currently planning to explore the region near $\beta = 6.4$ in detail, and in particular, pin down the location of the deconfining transition.

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[2]Throughout this paper the term scaling implies both (1) and (2). This is called asymptotic scaling in many recent preprints.

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