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Unambiguously Complete Characterization of Reactions

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It is shown that in order to determine all reaction amplitudes for a reaction containing particles with arbitrary spins without even a discrete ambiguity (except for an overall phase factor), one must measure at least one observable with spin polarization along each of three noncoplanar directions. The result has an impact on experimental programs measuring polarization quantities in elementary particle, nuclear, atomic, and molecular physics.

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The measurement of reaction amplitudes has occupied a central role in recent years in all branches of microscopic physics, because it has been realized that a reliable testing of dynamical theories can be performed only on such an amplitude level (instead of on the differential cross section level), because clues toward yet unknown dynamics are best obtained on such a level, and because the techniques for performing sophisticated polarization experiments have developed rapidly.

A central question in connection with such measurements of reaction amplitudes has been the characterization of complete sets of experiments that can determine the reaction amplitudes. The word "complete" has been used in two meanings. One denotes a set that can eliminate the entire *continuum* of ambiguities from the determination of amplitudes (except for an overall phase factor, common to all amplitudes, which is unobservable in a purely experimental or phenomenological context) while still permitting the existence of discrete ambiguities which arise from the bilinear nature of the relationship between observables and amplitudes. There has been a large number of contributions[1,2] in the literature to the formulation of criteria for choosing complete sets of amplitudes in this sense.

The other meaning of "complete" is the more literary or strict one, namely a set that eliminates *all* ambiguities, continuous or discrete (except again for the overall phase factor mentioned above). This problem has been much less discussed, even though in actual amplitude analyses, such remaining discrete ambiguities have caused problems.[3]

The aim of this note is, therefore, to offer a completely general theorem about the nature of "truly complete" sets of experiments, namely those which determine the amplitudes free of *any* ambiguities (except for the overall phase factor mentioned above). The theorem states that in order to determine all reaction amplitudes for a reaction containing particles with arbitrary spins without even a discrete ambiguity (except for an overall phase factor), one must measure at least one observable with spin polarization along each of three noncoplanar directions.

The theorem is an extension of the theorem in Section II of Ref. 2, which proved that for a complete set which nevertheless allows discrete ambiguities, we need two nonparallel directions. The conceptual content of that proof can be summarized in a way which is different from the language used for the proof in Ref. 2, but which will serve us better in the present extension of the theorem.

The most suitable formalism for this is one in which the relationship between the polarization observables and the bilinear combinations of amplitudes ("bicoms") is as simple as possible. Such a class of formalisms is the optimal formalism[4-6] in which the spin tensors used contain the fewest nonzero matrix elements compatible with Hermiticity. In that formalism, for a four-particle reaction $A+B\rightarrow C+D$ the amplitudes are denoted by $D(ca;db)$, where a, b, c, and the d are the spin projections of the spins of particles A, B, C, and D, respectively, with respect to some (specified) quantization directions. The observables then are denoted by

$$\mathscr{L}(uvH_p,UVH_P;\xi\omega H_q,\Xi\Omega H_Q),$$

where u and v are indices pertaining to particle A, U and V to particle B, ξ and ω to particle C, and Ξ and Ω to particle D, and where H can be either "Real" (R) or "Imaginary" (I), and $H_p=R$ implies $p=+1$, while $H_p=I$ implies $p=-1$. In this formalism the relationship between observables and amplitudes, for the most general case of a reaction with arbitrary spins, is given[4] by

$$\begin{aligned}\mathscr{L}(uvH_p,UVH_P;\xi\omega H_q,\Xi\Omega H_Q)\\ =\tfrac{1}{2}\kappa ZZ_2H_{wW}[&D(\xi u,\Xi U)D^*(\omega v,\Omega V)+\omega D(\omega v,\Xi U)D^*(\xi u,\Omega V)\\ &+pD(\xi v,\Xi U)D^*(\omega u,\Omega V)+pwD(\omega u,\Xi U)D^*(\xi v,\Omega V)\\ &+PD(\xi u,\Xi V)D^*(\omega v,\Omega U)+PwD(\omega v,\Xi V)D^*(\xi u,\Omega U)\\ &+pPD(\xi v,\Xi V)D^*(\omega u,\Omega U)+pPwD(\omega u,\Xi)D^*(\xi v,\Omega U)],\end{aligned}\qquad(1)$$

where $w=pq$, $W=PQ$, $Z_1=1+pq-p+q$, $Z_2=1+PQ-P+Q$, and $\kappa=1$ unless $w=W=-1$, in which case $\kappa=-1$.

It is evident from Eq. (1) that if all the H's on the left-hand side are R, we get on the right-hand side also R only. Thus we see that observables in which the four arguments are either diagonal (e.g., $u=v$, in which case H_p must be R) or off-diagonal and real (e.g., $u\neq v$, $H_p=R$) are linked only to bicoms which are the real parts of the products of two amplitudes. In order to find an observable which is linked to the imaginary part of the product of two amplitudes, we have to have (an odd number of) off-diagonal imaginary arguments in the observable.

The next step in our proof is to link the real and imaginary arguments in the observable with polarization directions of the particles. We know that, for arbitrary spin, the various polarization tensors for polarization in the direction of the quantization axis will be diagonal. Let us denote this direction by z. Then we also know that the vector polarization matrix in the x direction will be real and symmetric, and that in the y direction imaginary and antisymmetric. Higher polarization tensors in these directions will have the same property, since they can be built up out of these vector polarization matrices, and as $S_x^2+S_y^2+S_z^2=S(S+1)$, in this building up S_y needs to be used only linearly if we use S_x^2 and S_z^2. Thus we see that the polarizations in the x direction will correspond to R's in the arguments of the polarization observables, while the y directions will correspond to I.

Finally we note that if we determine only the absolute-value squares and the real parts of the products of amplitudes, discrete ambiguities must remain, since we then know only the cosines of the phase angles between amplitudes, and the inverse of the cosine is a double-valued function. In order to resolve such ambiguities, we must measure also some sines, that is, some observables with (an odd number of) I's in the arguments. This completes the proof.

Indeed, the proof suggests that if the reaction contains four particles with spins $s_1,\ldots,s_4$, and if we denote $x=\prod_{i=1}^4(2s_i+1)$, then the number of discrete sets (just after the *continuum* of ambiguities has been eliminated) could be as high as 2^{x-1}, and that by at most $x-1$ additional well-chosen experiments we can eliminate these discrete ambiguities. These numbers will be correspondingly reduced if the number of amplitudes is reduced by symmetry laws additional to Lorentz invariance.

We will now illustrate the result of the theorem on a very simple and familiar example, namely on the reaction $0+\frac{1}{2}\rightarrow 0+\frac{1}{2}$, with time-reversal invariance and parity conservation, which is realized, for example, in pion-nucleon elastic scattering. For that case we have only two reaction amplitudes which we will call α and β. Because this is the simplest nontrivial reaction imaginable, almost any of the conventional formalisms dealing with it are in fact optimal. For example, in the formalism in which the quantization axes of both spin-$\frac{1}{2}$ particles are normal to the reaction plane (i.e., the "transversity formalism"), the observables are related to the amplitudes in the following way:

$$d\sigma/d\Omega=|\alpha|^2+|\beta|^2,\quad P_n=A_n=|\alpha|^2-|\beta|^2,$$
$$K_{ll}=K_{ss}=\mathrm{Re}\alpha\beta^*,\quad K_{ls}=-K_{sl}=\mathrm{Im}\alpha\beta^*, \tag{2}$$

where n is the direction normal to the scattering plane, l and s are two mutually perpendicular directions in the scattering plane, P denotes simple polarization of the final particle, A denotes the asymmetry with a polarized initial particle, and K_{ab} is the polarization correlation with the initial particle polarized in the a direction and the final particle polarized in the b direction.

We see then by immediate inspection that the set $\{d\sigma/d\Omega, P_n, K_{ll}\}$ does not give an unambiguous determination of α and β, while any of the other three sets of three observables out of the four do. Note that $d\sigma/d\Omega$ (which is an average over the two quantization states) counts, in this frame, as an observable pertaining to polarization direction n. This completes the discussion of the example.

In special cases more detailed criteria exist for eliminating discrete ambiguities. For example, *if* we first determine, from a set of experiments, the *magnitudes* of all of the amplitudes, the necessary and sufficient criteria can be given[7] in terms of the bilinear products of amplitudes (but not necessarily in terms of the observables) for the completely unambiguous determination of the amplitudes. Whereas it is likely that such criteria will be made more general by future research, the result of this note will remain useful because it is so simple to apply, so general in its validity, and so directly applicable in designing experiments.

The consequences of this result for experimental programs can be spelled out as follows: If the aim is to eventually produce a truly unique set of amplitudes, arrangement should be made for the capacity to measure at least some polarization quantities in *all three* spatial directions. Furthermore, since ambiguities occur only in the relative phases of the amplitudes, and never in the magnitudes of the amplitudes, it is very advantageous to use an optical formalism for the description of the reaction in which the determination of the magnitudes of the amplitudes by themselves corresponds to a set of technologically easy experiments which then can be performed at the beginning of the experimental program. These magnitudes will then form a completely unambiguous initial set of partial information which, even in the absence of further experiments, can be used to assess theoretical proposals for dynamics. This initial set involves only one polarization direction. Adding the second direction will allow the determination of the phases up to discrete ambiguities (and, of course, up to an arbitrary overall phase factor). Finally, the addition of the third direction will choose also between the remaining discrete solutions. Thus we have outlined a three-stage structure for experimental programs, in which each stage provides functional information even in the absence of further stages, and in which the addition of further stages refines the already obtained information in a systematic way.

The theorem just presented complements results pertaining to polarization experiments along a different dimension, namely according to *how many particles* need to be polarized simultaneously. In that realm, we know that a complete determination of amplitudes can be made by a set of experiments each of which involves the simultaneous polarization of at most two particles. Another general theorem states that for a complete determination of the amplitudes, the set of particles in the reaction must *not* be divisible into two subsets such that no spin correlation between a particle in one subset and a particle in the other subset is measured at all. Our theorem places a requirement on a set of experiments which give a complete determination of the amplitudes along a different dimension, namely in terms of the number of different polarization *directions* that need to be represented in such a set.

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