Investigations on Scaling and Hyperscaling for Invasion Percolation

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Invasion-percolation observables which should scale with exponents interrelated through scaling and hyperscaling hypotheses are investigated. Evidence is presented that the hyperscaling relation between τ, the spatial dimension, and the fractal dimension breaks down at the defender threshold. Thus an independent exponent must be introduced to describe the scaling of the finite-defender-cluster distribution.

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Invasion percolation (IP) is a dynamic process in which a cluster grows into a sample through selection of paths of least resistance.[1-3] The resistances to invasion are assigned randomly to the sites (or bonds) of a regular lattice, and are held fixed throughout the process. The defender is treated as an "incompressible fluid": Once it has been surrounded the invader cannot penetrate it further. There can then exist two critical points in the process. The first, or "breakthrough," point is the point at which the invader first crosses the lattice. The second, "terminal," critical point occurs when the defender has been completely disconnected into finite clusters; for this to happen it is essential that a "semipermeable wall" enclose the system, so that the defender may escape the region but the invader may not. While in $d=2$ dimensions the terminal point is identical to the breakthrough point,[3] for $d \geqslant 3$ the two critical points are substantially different.

It is useful to analyze IP by drawing analogies with ordinary percolation. For example, the fraction of sites occupied by the invader at breakthrough, $S(L)$, scales as[1,3] $S(L)=S_0 L^{-(d-D_I)}$, where L is the linear size of the lattice in units of the lattice spacing, and D_I is the invader fractal dimension. Further, there is an independent exponent analogous to a thermal exponent which describes the behavior of the acceptance fraction of invaded sites.[2,3]

For ordinary percolation a powerful scaling formalism exists which relates scaling exponents of observables to two independent exponents.[4] However, no such formalism exists for IP, so that no "proofs" of relationships among IP exponents exist at present. But as was noted in Ref. 3, one may hunt about and discover relations intrinsic to the model (purely geometrical) which are satisfied at each of the critical points. In this Letter I report on several such relationships for geometric exponents in IP.

The present results indicate that at breakthrough the fractal dimension is sufficient to quantify IP scaling observables, as in ordinary percolation. But at the terminal critical point it appears necessary to introduce a new independent exponent. It was reported in Ref. 3 that at the terminal threshold the number of defenders in clusters containing s sites, $n(s)$, scales with s according to the power law $n(s) \approx s^{-\tau}$. (The exponent τ is meaningless for IP at breakthrough for $d=3$.) With the present data set, I find a least-squares fit[5] $\tau = 2.05 \pm 0.04$. Now, in ordinary percolation the relation $\tau = 1 + d/D_I$ follows from hyperscaling.[4] To test such a relation for IP, I introduce a defender fractal dimension D_s at the terminal point, and find that $D_s \neq D_I$. I then ask if τ satisfies the hyperscaling relation using D_s. The relation might hold because it encapsulates a statement about how the finite clusters are distributed in space, i.e., self-similarity under scale transformations in the precise way to be described in Eq. (3). But I find the the relation does not hold, and thus learn that the self-similarity of the clusters occurs with a characteristic dimension different from the spatial dimension.

We now discuss breakthrough in detail. As is conventional,[6] introduce the local density $\rho(\vec{x})=1$ if $\vec{x}$ is occupied by the invader; $\rho(\vec{x})=0$ otherwise. The scaling of the invader saturation suggests we assign the scaling dimension $D_I - d$ to the field ρ. Then the autocorrelation function

$$C(\vec{r},L)=L^{-d}\textstyle\sum_{\vec{x}}\rho(\vec{x})\rho(\vec{x}+\vec{r})$$

should scale as $C(r,L)=L^{-2(d-D_I)}f(r/L)$ for $L > r >> 1$. The replacement of $\vec{r}$ by $r=|\vec{r}|$ follows from $xy(z)$ invariance, which is easily verified in the model. (This invariance basically reflects the statistical isotropy of laying the random numbers on the sites of the lattice at the outset.)

Next, introduce the conditional probability that a site a distance r away from an occupied site is also occupied, $\sigma(r,L)=C(r,L)/S(L)$. From this construct the "partial saturations"

$$S(r,L)=r^{-d}\int_0^r dx\, x^{(d-1)}\sigma(x,L). \qquad (1)$$

We shall say that an invader cluster is a convention-

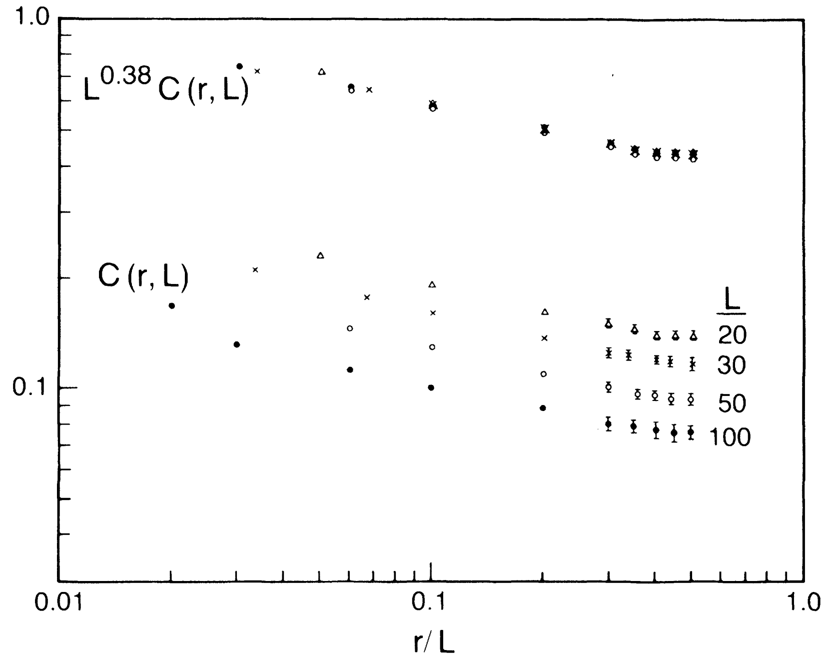

FIG. 1. Correlation function for $d=2$. The lower set of graphs correspond to $L=20$, number of realizations $R=500$; $L=30$, $R=300$; $L=50$, $R=200$; and $L=100$, $R=200$. The upper curve indicates the degree to which scaling holds for this correlation function.

al fractal if $S(r,L) \approx r^{-(d-D_I)}$ for $L >> r >> 1$.

The above functions have been examined for $d=2,3$ with use of square and simple-cubic lattices. The initial conditions chosen correspond to an experimental configuration of interest: The invader penetrates a sample longer than it is wide (width $=L$) from one face until it reaches the far face. The function $\rho(\vec{x})$ is then defined for points $\vec{x}$ in an interior region L^d of the sample, in order to minimize transient effects. The number of realizations averaged over is indicated in the figure captions.

Consider first $C(r,L)$ for $d=2$. Figure 1 displays $C(r,L)$ for several L values. The product $L^{2(2-D_I)}C(r,L)$ is also plotted in this figure, and we see that the scaling hypothesis is supported by the simulation data over a significant range in r/L. Of course, for $r \to 0$, the scaling hypothesis cannot apply since $C(r=0,L)=S(L)$. Notice also that as $r \to L/2$, $C(r,L) \to S^2(L)$. The line of data points at the top of the figure follows $f(r/L) \approx (r/L)^{-0.19}$, in agreement with the reported[2] value $D_I=1.81$.

Consider next the partial saturations $S(r,L)$. First, for $r=L$, we have

$$S(L,L) \approx L^{-(2-D_I)} \int_0^1 dx\, x f(x)$$

which scales with the same power as $S(L)$. But for r in the scaling region, I expect $S(L >> r >> 1,L) \approx r^{-0.19}$, independent of L. Figure 2 exhibits simulation results for $S(r,L)$. Any L dependence other than an overall dependence governing the onset of the scaling regime is certainly very small. Further, $D_I=1.81$ fits the data reasonably well in the (admittedly narrow) scaling regime.

Passing to $d=3$, I find that $C(r,L)$ and $S(r,L)$ behave qualitatively just as in $d=2$, with different D_I in the two dimensionalities. Detailed analysis of the data will not be recorded here, as it contains no surprises. Insofar as growth to breakthrough is concerned, then, the fractal description of the growing cluster is self-consistent in both $d=2$ and $d=3$. Equivalently, scaling hypotheses work for the systems.

We now turn to the terminal critical point in $d=3$. The finite clusters with s sufficiently large that they are in the scaling region of $n(s)$ can be considered fractals in the following sense:

$$s = R(s)^{D_s}, \tag{2}$$

where $R(s)$ is the radius of gyration of the cluster. Figure 3 exhibits data on Eq. (2). These data indicate a value $D_s = 2.13 \pm 0.05$, which is not equal, within the errors, to $D_I = 2.47$ for $d=3$.

In ordinary percolation, there exists a single exponent $D = D_s = D_I$. Recall that the exponent τ is related to D by the equation $\tau = 1 + d/D$. Since we

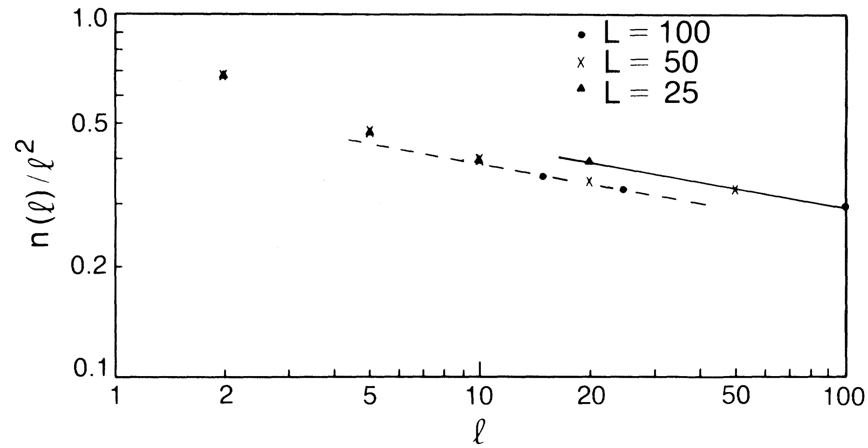

FIG. 2. Fractal dimension by counting sites in increasingly larger boxes of side l inside a lattice of side L for $d=2$. The ordinate is the density of sites at size l. The solid line indicates how the overall fraction of invader scales as a function of L.

have verified that the s-independent prefactor in $n(s)$ is L independent, we can combine this equation with the power law for $n(s)$ and conclude that

$$\lambda \frac{d}{d\lambda}[L(\lambda)^d s(\lambda) n(s(\lambda))]=0. \tag{3}$$

The functional dependences upon λ exhibited above mean that if $L \to \lambda L \equiv L(\lambda)$, $s \to \lambda^{D_s} s \equiv s(\lambda)$, consistently with Eq. (2).

Equation (3) states that the spatial distribution of s-clusters is self-similar under scale transformations. It is useful to think of scaling $L \to \lambda L$ as an increase of magnification of a microscope used to examine the clusters. A cluster which at scale L had s sites is revealed to have $s(\lambda)$ sites at increased magnification for fixed resolution of our microscope. Nonetheless, Eq. (3) informs us that, on average, the total number of sites in s-clusters contained in a box of size L, $N(s,L)=L^d sn(s)$, is invariant.

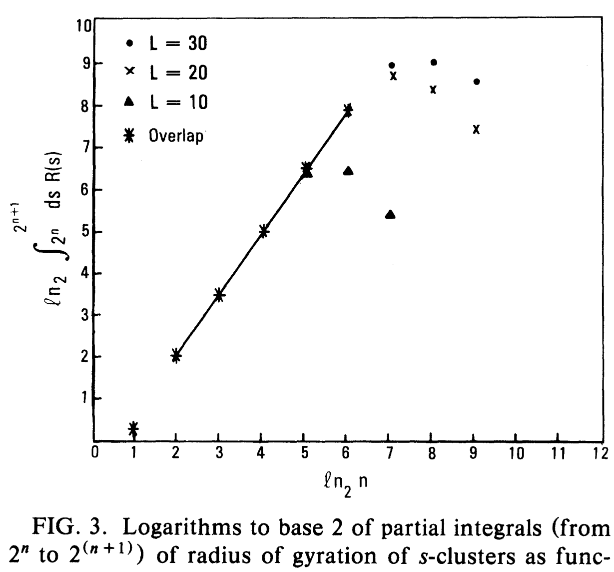

FIG. 3. Logarithms to base 2 of partial integrals (from 2^n to $2^{(n+1)}$) of radius of gyration of s-clusters as functions of $\log_2 s$ $(=n)$. Data are for $d=3$, $L=30$, number of realizations $R=50$; $L=20$, $R=100$; and $L=10$, $R=100$.

But the values for τ and D_s in IP do not satisfy the equation $\tau = 1 + d/D_s$, which led to Eq. (3). So consider the alternative equation

$$\tau = 1 + d'/D_s. \tag{4}$$

A modified version of Eq. (3) holds as a consequence of Eqs. (2), (4), and the power law for $n(s)$, in which the term $L(\lambda)^d$ is replaced by $L(\lambda)^{d'}$. The present results are best fitted by $d' = 2.24 \pm 0.10$. What is to be made of this? The modified Eq. (3) implies that $N(s,L)$ increases under magnification λ by a factor $\lambda^{(d-d')}$ (see Fig. 4).

How can it happen that $N(s,L)$ changes under magnification? In principle it is always possible that clusters which were connected at one magnification will consist of disconnected parts at higher magnification. Thus clusters of any s in the scaling region will lose some of their number to clusters of smaller s, but also gain in number through the breakup of larger clusters. Evidently, when $d' = d$, the competition between loss and gain balances out exactly. For $d' < d$, there is a net "cascade" of larger clusters into smaller ones. Plausible consequences of this cascade are an ever earlier onset of the scaling regime for s-clusters as the sample size increases, accompanied by sharp depletion of the very large clusters. Although both tendencies are evident in

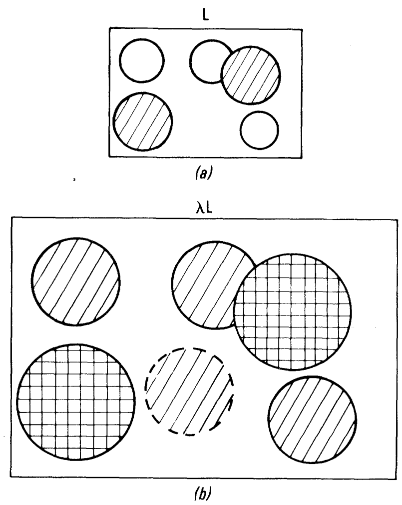

FIG. 4. Self-similarity of distribution of s-clusters. In (a) clusters of size s_1 are depicted in terms of their radii of gyration, open circles. The hatched circles depict clusters of size $s_2 = \lambda^{D_s} s_1$. Thus at magnification λ, we see in (b) that the hatched circles, corresponding to $s_1(\lambda)$, actually contain s_2 sites. Meanwhile the old s_2 clusters have expanded into $s_2(\lambda)$ clusters, depicted in double-hatch. The new hatched circle with the dotted perimeter represents the net new clusters of s_2 sites which appear upon magnification when $d' < d$.

the present data, I do not claim that this constitutes additional evidence for this picture.

Because of the cost in time of invasion simulations, we must deal with rather small statistics. To what extent do the observations depend upon these statistics, and upon the algorithms for computing the quantities involved? To test the algorithms "random" clusters have been created by filling sites randomly in a lattice of $L = 30$ to an occupation fraction of 0.353, which is the average defender occupation at the end of IP on a lattice of this size. I then compute the exponents τ and D_s which characterize a purely random site selection process. It is found that D_s is indistinguishable from the value obtained in IP. However, the exponent $\tau_{\rm random} = 2.39 \pm 0.05$ is significantly different,[7] and leads to $d'_{\rm random} = 2.96 \pm 0.13$. The result that D_s is approximately the same in the two kinds of simulation is disturbing, but may relate to the use of Eq. (2) to define D_s. On the other hand, Eqs. (2)–(4) are internally consistent, and the observed self-similarity with dimension close to 3 for random occupation but close to 2 for IP is numerically significant.

In conclusion, the existence of distinct fractal dimensions at the distinct critical points of IP is not necessarily surprising. What is new and interesting is that the specific correlations built into selecting the finite clusters at the terminal point of IP alter the "hyperscaling dimension" from the spatial dimension to a new fractal value.

I have benefited from discussions with R. Dashen, L. Kadanoff, and D. Wilkinson. Special thanks are given to D. W. for use of his IP simulation program.

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[4]See, e.g., D. Stauffer, Phys. Rep. **54**, 1 (1979).

[5]See, e.g., D. C. Baird, *Experimentation: An Introduction to Measurement Theory and Experiment Design* (Prentice-Hall, Englewood Cliffs, 1962).

[6]See, e.g., A. Kapitulnik *et al.*, J. Phys. A **16**, L269 (1983).

[7]Notice that the "random" defenders are above the random percolation threshold 0.3115. Thus the exponents for these need not be identical to those for ordinary percolation. Keep in mind that the purpose of the exercise is to check that the algorithm successfully detects differences between the IP clusters and the random clusters.