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General Theory for Quantum Statistics in Two Dimensions

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Because of complicated topology of the configuration space for indistinguishable particles in two dimensions, Feynman's path-integral formulation allows exotic statistics. All possible quantum statistics in two-space are characterized by an angle parameter θ which interpolates between bosons and fermions. The current formalisms in terms of topological action of multivalued wave functions can be derived in a model-independent way.

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Physics in two spatial dimensions is not always simpler than physics in three or higher dimensions. By now the well-known examples include the appearance of both fractional angular momentum or spin[1-3] and exotic statistics.[4-7] Both of them are forbidden in space of dimension $\geqslant 3$; in fact, they are related to special topological situations in two-space.[8] The physical relevance of these exotic statistics is expected to be in condensed matter physics where two-dimensional systems have become experimentally available.[9] Since a wide variety of model Hamiltonians can be constructed for various actual materials, it is important to know how generally true are the results and conclusions obtained in the recent study[5-7] of exotic statistics in two concrete models, namely the flux-tube-charge composites[1] and topological solitons.[2] In particular, are there more exotic statistics other than those found in these models? This Letter is devoted to discussing these questions.

I will work in Feynman's path-integral formalism of quantum statistics.[10] The propagator for a system is a sum over all continuous paths in the configuration space connecting the initial state q and the final state q'. Since the configuration space of n indistinguishable particles, M_n, is always multiply connected (see below), paths in different homotopy classes cannot be continuously deformed into each other. Thus the propagator is actually a (weighted) sum over "partial amplitudes," each being an integration over paths belonging to a distinct homotopy class:

$$K(q't';qt) = \sum_{\alpha \in \pi_1} \chi(\alpha) \int_{q(t) \in \alpha} \exp\{i \int_q^{q'} dtL\} \mathscr{D}q(t). \quad (1)$$

The homotopy classes α of paths from q to q' can be made to be identified with the elements of $\pi_1(M_n)$, the fundamental group of M_n, by choosing a mesh of "standard paths" from a fixed point q_0 to every point in M_n and adjoining the path qq' to the standard ones q_0q and $q'q_0$ to form a loop. The point here is that, as in quantum mechanics on any multiply connected space,[11] the complex weights $\chi(\alpha)$ for different homotopy classes have no *a priori* reason to be the same. Invariance under different choices of the standard path mesh and the composition law for the propagator require that $\chi(\alpha)$ must be a phase factor and form a representation of $\pi_1(M_n)$.[10]

What is M_n? For distinguishable particles, the configuration space is just $R^{dn} \equiv R^d \times \cdots \times R^d$ (n factors), where d is the dimension of space. For indistinguishable particles, there is no physical distinction between points in R^{dn} which differ from each other only in the ordering of particle indices. So the symmetric points in R^{dn} (under the action of the symmetric group of n objects, S_n) should be identified [e.g., $(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n) \sim (\vec{r}_2, \vec{r}_1, \cdots, \vec{r}_n)$]. Because we are not always guaranteed that there is a finite probability for two particles to coincide with each other, the so-called diagonal points in R^{dn}, $D = \{(\vec{r}_1, \cdots, \vec{r}_n)$ with $\vec{r}_i = \vec{r}_j$ for some $i \neq j\}$, have to be excluded too.[12] Thus the configuration space of n indistinguishable particles is $M_n = (R^{dn} - D)/S_n$.

For $d \geqslant 3$, $\pi_1(R^{dn} - D) = 0$, so that $\pi_1(M_n) = S_n$. There are only two one-dimensional representations of S_n: $\chi_+(\alpha) = 1$ for all α and $\chi_-(\alpha) = \pm 1$ according as α is an even or odd permutation. The physical meaning of $\chi(\alpha)$ is now clear: it determines the statistics. In three or higher dimensions there are only two kinds of statistics, either Bose-Einstein (with χ_+) or Fermi-Dirac (with χ_-), with no possibility for exotic ones in a path-integral formulation.[10]

In two dimensions $\pi_1(M_n)$ is much more complicated; it is an infinite non-Abelian group. Fortunately, its structure has been clarified for some time.[13] I give a pictorial illustration as follows. Recall that a closed path in M_n can be represented by n curves in the three-space (x,y,t) with no intersections and with the final positions in R^2 at t' being just permutations of the initial ones at t. I display the equivalence classes of these curves by projecting them on a fixed x-t plane. To distinguish, the projections on the plane will be called strings. Without loss of generality, we can assume that (1) the initial positions of the strings are all different (i.e., $x_1 < \cdots < x_n$), (2) at each time slice there is at most one intersection of two neighboring strings, and (3) the strings are always parallel to the t axis, with x values being permuted initial ones, except in the neighborhood of an intersection. To keep track of how the curves in three-space wind, we let one of the strings at the intersection be in front if the corresponding curve in three-space has smaller ordinate at that point. Such a configuration of strings is called a braid.[14] (Some examples are shown in Figs. 1 and 2.) The multiplication of two braids follows from that of two closed paths in M_n, so that the equivalence classes of braids under continuous deformation also form a group, called the braid group, $B_n(R^2)$. From what is said above, it is isomorphic to $\pi_1(M_n)$.

Some features of the braid group are easily recognized. Denote by σ_i the operation of interchanging two neighboring strings at x_i and x_{i+1} with the left one in front. Then a braid can be algebraically expressed as a product of a sequence of $\sigma_i^{\pm 1}$ $(1 \leqslant i \leqslant n-1)$. From Figs. 1 and 2 it follows

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},$$
$$\sigma_1 \sigma_k = \sigma_k \sigma_i \quad (k \neq i \pm 1). \tag{2}$$

There are no further relations among σ_i's.[15]

From Eq. (2), all one-dimensional unitary representations of $\pi_1(M_n)$ satisfy

$$\chi_\theta(\sigma_1) = \cdots = \chi_\theta(\sigma_{n-1}) = e^{-i\theta}. \tag{3}$$

and are labeled by θ $(0 \leqslant \theta < 2\pi)$. As a natural generalization of the interpretation of $\chi(\alpha)$ from three to two dimensions, the $\chi_\theta(\alpha)$ in Eq. (1) represent new types of statistics, which we call θ statistics. It interpolates the Bose-Einstein $(\theta = 0)$ and Fermi-Dirac $(\theta = \pi)$ statistics. (An example for such interpolation is known in one dimension.[16])

To have more explicit understanding of θ statistics, we need more knowledge of $\chi_\theta(\alpha)$. $\alpha \in \pi_1(M_n)$ is always a product of a sequence of $\sigma_k^{\pm 1}$. Physically, σ_k represents interchange of only the two particles at $\vec{r}_k$ and $\vec{r}_{k+1}$ along a counterclockwise loop with other particles kept outside. Since we label particles by their initial positions and the particles temporarily at $\vec{r}_k$ and $\vec{r}_{k+1}$ can be any two of them, we can rewrite Eq. (3) as

$$\chi_\theta(\sigma_k^{\pm 1}) = e^{\mp i\theta} = \exp\{-i(\theta/\pi) \sum_{i<j} \Delta\phi_{ij}\}, \tag{4}$$

where $\Delta\phi_{ij}$ is the change of the azimuthal angle of

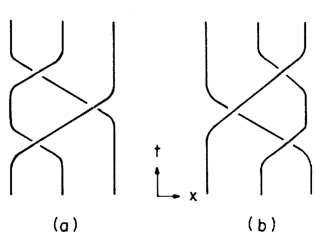

FIG. 1 Two braids for $n = 3$.

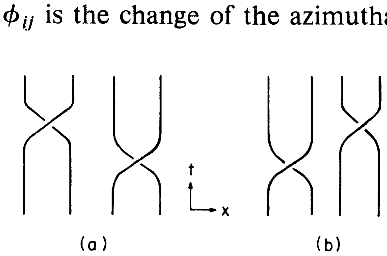

FIG. 2. Two braids for $n = 4$.

particle i relative to particle j. For each σ_k, only one term in the sum is nonvanishing and its value is π. This formula can be easily generalized to arbitrary $\alpha \in \pi_1(M_n)$:

$$\chi_\theta(\alpha) = \exp\left\{-i\frac{\theta}{\pi}\int dt \frac{d}{dt}\sum_{i<j}\phi_{ij}(t)\right\}. \tag{5}$$

Note that the right-hand side is indeed a homotopic invariant.

When we insert Eq. (5) into Eq. (1), the right-hand side of Eq. (5) includes also the contributions from the standard paths $q_0 q$ and $q' q_0$. However, they contribute only an overall phase factor to K so that we can neglect them:

$$K(q't';qt) = \int \exp\left\{i\int_q^{q'} dt\left[L - \frac{\theta}{\pi}\frac{d}{dt}\sum_{i<j}\phi_{ij}(t)\right]\right\}\mathscr{D}q(t). \tag{6}$$

Here $q(t)$ is a path in $R^{dn} - D$, but paths with initial particle positions differing only in permutations should be included. Thus, the inclusion of $\chi_\theta(\alpha)$ is equivalent to addition of a topological action which does not affect the equations of motion but determines statistics.[6]

There is another way to eliminate the phase factors $\chi(\alpha)$ and the sum over α. Let us consider the set, $\tilde{M}_n$, of all equivalence classes of paths in M_n with a given final point q_0. Paths with different initial points are necessarily in different classes. Mathematically, $\tilde{M}_n$ is identified with the universal covering space of M_n. Closed loops through a point q in M_n in different homotopy classes can be viewed as open paths in $\tilde{M}_n$ from point $\tilde{q}$ to the corresponding points $\tilde{q}\alpha$ on different sheets. [We write the action of $\pi_1(M_n)$ on $\tilde{M}_n$ to the right.] Thus the path integral in Eq. (1) over $q(t) \in \alpha$ in M_n can be viewed as a propagator in $\tilde{M}_n$ from $\tilde{q}\alpha^{-1}$ to $\tilde{q}'$ corresponding to the original Lagrangian L:

$$\tilde{K}(\tilde{q}'t';\tilde{q}\alpha^{-1},t) = \int \exp\{i\int_{\tilde{q}\alpha^{-1}}^{\tilde{q}'} dtL\}\mathscr{D}q(t). \tag{7}$$

Now from the wave function $\psi(q,t)$, which is single valued in M_n and propagates according to K, i.e.,

$$\psi(q',t') = \int_{M_n} dq\, K(q't',qt)\psi(q,t), \tag{8}$$

we can define a new wave function $\tilde{\psi}(\tilde{q},t)$ in $\tilde{M}_n$:

$$\tilde{\psi}(\tilde{q},t) = \exp\{-i(\theta/\pi)\int_q^{q_0} d(\sum_{i<j}\phi_{ij})\}\psi(q,t), \tag{9}$$

where the integral is along a path in M_n which is identified with the point $\tilde{q}$ in $\tilde{M}_n$. It is single valued in $\tilde{M}_n$ and its propagation obeys

$$\tilde{\psi}(\tilde{q}',t') = \int_{\tilde{M}_n} d\tilde{q}\, \tilde{K}(\tilde{q}'t',\tilde{q}t)\tilde{\psi}(\tilde{q},t) \tag{10}$$

since the phase factor in Eq (9) is chosen in accordance to Eq. (5). As

$$\tilde{\psi}(\tilde{q}\alpha,t) = \chi(\alpha^{-1})\tilde{\psi}(\tilde{q},t) \tag{11}$$

by identifying all the points $\tilde{q}\alpha$ with q, $\tilde{\psi}$ can be also considered as a wave function (though multivalued) in M_n. Nothing is wrong with this multivalued wave function, because all branches have the same modulus, and the multivalued phase factors are just right to keep track of the weights $\chi(\alpha)$.

Equation (9) can be rewritten in terms of complex coordinates z_i:

$$\tilde{\psi}(z_i,z_i^*;t) = \prod_{i<j}(z_i - z_j)^{\theta/\pi} f(z_i,z_i^*;t) \tag{12}$$

with f single valued and symmetric in pairs of (z_i,z_i^*). This $\tilde{\psi}$ will satisfy the ordinary Schrödinger equation without the extra θ-dependent term. When $n = 2$, an exchange of the particle positions gives rise to a phase factor $e^{im\theta}$, m being the winding number. As emphasized previously,[7] for a system having three or more particles, Eq. (12) exhibits even more complicated behavior under interchange of particle positions.

Thus θ statistics can be considered either as a natural generalization of normal statistics in which θ appears in the phase that the wave function acquires under exchange of particles, or as due to a peculiar long-range interaction arising from a topological action where θ appears as a coupling constant. In the first way, the notion of wave functions must be generalized.

To conclude, some remarks are in order. First, Eqs. (6) and (12) have already appeared in Refs.

5–7. So all the results and conclusions derived there from these equations are generally true. Especially the θ statistics in those models exhaust all exotic possibilities.

Second, mathematically there is a very close analogy[5] of all this to θ worlds[17] in gauge theories. In a path-integral formulation in the gauge $A_0=0$, the configuration space of a non-Abelian gauge theory is the quotient space $\mathscr{A}/\mathscr{G}$,[18] where $\mathscr{A}$ is the space of gauge potentials in three-space and $\mathscr{G}$ is the group of gauge transformations in three-space. Since $\pi_i(\mathscr{A}/\mathscr{G})=\pi_0(\mathscr{G})=\pi_3(G)=Z$, all one-dimensional unitary representations of $\pi_1(\mathscr{A}/\mathscr{G})$ are characterized by an angle parameter θ too. The "vacuum angle" θ appears either as a coupling constant in a topological action or in the multivalued phase of a wave functional in $\mathscr{A}/\mathscr{G}$ which is single valued in the universal covering space $\mathscr{A}$. I emphasize this parallel to show that nothing is ill defined or mysterious with θ statistics.

Finally, since the treatment is model independent, we expect the appearance of θ statistics in two dimensions on a general ground. It is worthwhile looking for the signal for it in two-dimensional physical systems.

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