PHYSICAL REVIEW LETTERS

VOLUME 52 | 4 JUNE 1984 | NUMBER 23

Classical Model of the Dirac Electron

A. O. Barut
Department of Physics, The University of Colorado, Boulder, Colorado 80309

and

Nino Zanghi
Department of Mathematics, The University of Denver, Denver, Colorado 80208
(Received 1 March 1984)

A covariant, symplectic, classical dynamical system is presented whose quantization, by replacement of the Poisson brackets with commutators, gives precisely the Dirac electron theory. For the classical system the velocity and momentum are independent dynamical variables, as in Dirac electron; it undergoes a *real Zitterbewegung* so that the spin is the orbital angular momentum of the *Zitterbewegung*. Thus the longstanding problem of representing the configuration space of quantum spin by a classical model is solved. Three different forms of the dynamical equations are given.

PACS numbers: 03.20.+i, 11.10.Qr, 14.60.Cd

Historically quantum spin was introduced as a "classically nondescribable two-valuedness," as Pauli described it in connection with his exclusion principle.[1] Ever since, many attempts to model the electron as some kind of spinning top failed, simply because the configuration space of the quantum spin is quite different from that of a top.[2] We present here a classical relativistic model of an electron with internal degrees of freedom which are different than those of a top but in precise correspondence with the quantum system, and whose quantization exactly gives the Dirac electron. In the process we verify for the classical model the intuitive picture of the quantum spin proposed by Schrödinger a long time ago,[3] called "*Zitterbewegung.*"

It is well known that the Dirac equation in Heisenberg representation has three sets of *independent* dynamical variables: position of the charge $\vec{x}$, velocity of the charge $\dot{\vec{x}} = c\vec{\alpha}$, and momentum $\vec{p}$. Position and velocity, or momentum and velocity, can be specified simultaneously, they commute, but position $\vec{x}$ and momentum $\vec{p}$ do not. This gives rise to the remarkable phenomenon of *Zitterbewegung,*[3] which is the rapid oscillatory motion of the charge with velocity c around a center of mass that is moving like a relativistic particle with velocity $\vec{p}/m$. The new center of mass and relative coordinates are defined in terms of the above three sets of dynamical variables, and one can separate the internal and external algebras and obtain a new type of geometry of the phase space.[4] A third way of expressing dynamical variables is via the spin variables, and the spin appears as the orbital angular momentum of the *Zitterbewegung.*[5]

All of these statements are made, however, in terms of Heisenberg operators. In order to really visualize *Zitterbewegung* one has to take expectation values with wave packets. And if one takes expectation values between purely positive energy states it is well known that *Zitterbewegung* disappears.

But the Dirac particle is a rather different kind of dynamical system than a point particle with an (x,p) phase space only, because of the additional dynamical variables; in fact it is more like the radiating classical Lorentz-Dirac electron including radiation reaction.[6] Therefore, it is important to find the classical symplectic system corresponding to it. Some time ago, Grossmann and Peres[7] went backwards, turning the Dirac commutators into Poisson brackets. They had some difficulties with manifest covariance and the physical interpretation of the proper time and of some of the dynamical variables. In this work, we start directly with the Lagrangian and Hamiltonian of the classical system in manifestly covariant form and derive the different forms of the dynamical equations. The invariant time variable turns out to be the proper time of the center of mass, and not of the charge, which removes the previous difficulties. The set of dynamical variables can be choosen both in classical and in quantum theory in three different ways: (i) the center of mass and relative coordinates and momentum; (ii) the coordinates of the charge x, momentum p, and spin variables; or (iii) the coordinates and momentum of the charge, and the internal oscillating degrees of freedom.

Furthermore, we shall also show that the Dirac electron in Heisenberg representation, the classical model, and the classical Lorentz-Dirac radiating electron all satisfy a third-order equation in x of exactly the same form; this is a fourth way of representing the additional spin degrees of motion.

The classical system is characterized by the usual pair of conjugate variables (x_μ, p_μ) and by another pair of conjugate classical spinor variables $(z, -i\bar{z})$ representing internal degrees of freedom. The configuration space is thus $M_4 \times C_4$. We shall use an invariant time variable τ. The motion in $M_4 \times C_4$ when projected down to M_4 will look like a particle with internal spin degrees of freedom. Here $z \in C_4$ is a Dirac spinor and $\bar{z} = z^\dagger \gamma^0$.

The Lagrangian is given by

$$L = \tfrac{1}{2}\lambda i(\dot{\bar{z}}z - \bar{z}\dot{z}) + p_\mu(\dot{x}^\mu - \bar{z}\gamma^\mu z) + eA_\mu(x)\bar{z}\gamma^\mu z, \tag{1}$$

where λ is a constant with the dimension of action ($c=1$). We can view here p_μ as Lagrange multipliers when the velocities $\dot{x}_\mu$ are represented by $\bar{z}\gamma^\mu z$. This representation of velocities was first used by Proca[8,9] who considered, however, only solutions of (1) corresponding to spinless particles. The structure of the system (1) is much richer, as we shall see. The dynamical variables are $(z, \bar{z}, p_\mu, x_\mu)$ and the Euler-Lagrange equations corresponding to (1) are as follows (I):

$$\begin{gathered} \dot{z} = i\pi z, \quad \dot{\bar{z}} = -i\bar{z}\pi, \\ \dot{\pi}_\mu = eF_{\mu\sigma}v^\sigma, \quad v_\sigma = \bar{z}\gamma_\sigma z = \dot{x}_\sigma, \end{gathered} \tag{2}$$

with

$$\begin{gathered} \pi \equiv \gamma^\mu \pi_\mu, \quad \pi_\mu = p_\mu - eA_\mu, \\ F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \end{gathered} \tag{3}$$

We note right away that $\dot{x}_\mu \dot{x}^\mu \neq 1$. Thus the invariant parameter τ cannot be interpreted as the proper time of the charge. We shall later see that τ is precisely the proper time of a "center of mass" which will be defined.

For a free particle the solutions of Eqs. (2) can be found easily. They are (with $\lambda = 1$)

$$\begin{gathered} z(\tau) = [\cos m\tau + (\gamma^\mu p_\mu/m)\sin m\tau]z(0), \\ \bar{z}(\tau) = \bar{z}(0)[\cos m\tau - i(\gamma^\mu p_\mu/m)\sin m\tau], \\ v_\mu = \dot{x}_\mu = \frac{p_\mu}{m^2}H + \left[\dot{x}_\mu(0) - \frac{p_\mu}{m}H\right]\cos 2m\tau + \frac{\dot{x}_\mu(0)}{2m}\sin 2m\tau, \\ p_\mu = \text{const}, \quad p^2 = m^2, \quad H = \dot{x}_\mu p^\mu. \end{gathered} \tag{4}$$

Quite generally, for the system (2), $\mathscr{H} = \dot{x}^\mu \pi_\mu$ is a constant of motion. For a free particle, $\mathscr{H} = H = m$ defines the parameter m which we can identify with mass m [which does not enter Eqs. (2)]. In the solutions (4) we already see the classical analog of the phenomenon of *Zitterbewegung.* With $H = m$, the velocity $\dot{x}_\mu = v_\mu$ has a term p_μ/m as it should, plus an oscillating motion with the characteristic frequency $\omega = 2m$.

Instead of the variables z and $\bar{z}$ we can work in terms of the spin variables. The set (2) is equivalent to the following closed set of dynamical equations (II):

$$\begin{gathered} \dot{x}_\mu = v_\mu, \quad \dot{v}_\mu = 4S_{\mu\rho}\pi^\rho, \\ \dot{\pi}_\mu = eF_{\mu\rho}v^\rho, \quad \dot{S}_{\mu\nu} = \pi_\mu v_\nu - \pi_\nu v_\mu \end{gathered} \tag{5}$$

for the set of dynamical variables $(x_\mu, v_\mu, \pi_\mu, S_{\mu\nu})$. The connection between z, $\bar{z}$, and $S_{\mu\nu}$ is

$$S_{\mu\nu} = \tfrac{1}{4}i\bar{z}[\gamma_\mu, \gamma_\nu]z. \tag{6}$$

The Hamiltonian form of the equation of motion and the Poisson brackets are defined with respect to the constant of motion $\mathscr{H} = \pi_\lambda \bar{z}\gamma^\lambda z$ which is the "Hamiltonian" with respect to the invariant parameter τ. In terms of the conjugate pairs of variables

(x,p) and $(z,i\bar{z})$, the basic Poisson brackets are

$$\{f,g\} \equiv i\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{z}} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar{z}}\right) + g^{\mu\nu}\left(\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial p^\nu} - \frac{\partial f}{\partial p^\mu}\frac{\partial g}{\partial x^\nu}\right), \tag{7}$$

$$\{x_\mu,x_\nu\}=0, \quad \{x_\mu,\pi_\nu\}=g_{\mu\nu}, \quad \{\pi_\mu,\pi_\nu\}=eF_{\mu\nu}.$$

We then obtain the Poisson brackets for $v_\mu,S_{\mu\nu}$ as follows:

$$\{v_\mu,v_\nu\}=4S_{\mu\nu}, \quad \{S_{\alpha\beta},v_\gamma\}=g_{\alpha\gamma}v_\beta-g_{\beta\gamma}v_\alpha,$$
$$\{S_{\alpha\beta},S_{\gamma\delta}\}=g_{\alpha\gamma}S_{\beta\delta}-g_{\beta\gamma}S_{\alpha\delta}-g_{\alpha\delta}S_{\beta\gamma}+g_{\beta\delta}S_{\alpha\gamma}. \tag{8}$$

Clearly, the set (5) is equivalent to the Poisson bracket of $y=(x_\mu,v_\mu,p_\mu,S_{\mu\nu})$ with $\mathscr{H}$, i.e., $\dot{y}=\{y,\mathscr{H}\}$.

In order to study the structure of the internal dynamics of the particle, we split x_μ and $\dot{x}_\mu=v_\mu$ as follows:

$$x_\mu=X_\mu+Q_\mu, \quad v_\mu=V_\mu+U_\mu \tag{9}$$

in such a way that by definition, $V^\mu=\dot{X}^\mu$. For a free particle, $\dot{V}_\mu=0$ and $V^\mu=p_\mu/m$. We can interpret X_μ and p_μ as the center-of-mass coordinates and Q_μ and $P_\mu=mU_\mu$ as the relative position and momentum. For a free particle, again, the dynamical system in terms of this new set is as follows (III):

$$\dot{X}_\mu=V_\mu, \quad \dot{V}_\mu=0,$$
$$\dot{Q}_\mu=m^{-1}P_\mu, \quad \dot{P}_\mu=-4m^3Q_\mu. \tag{10}$$

In particular, the last two equations show that the internal variables are oscillator coordinates,

$$\ddot{Q}_\mu+4m^2Q_\mu=0, \quad \ddot{P}_\mu+4m^2P_\mu=0, \tag{11}$$

with a frequency $2m$.

The orbital angular momentum $L_{\mu\nu}=x_\mu p_\nu-x_\nu p_\mu$ is not a constant of motion, but $\dot{L}_{\mu\nu}=v_\mu p_\nu-v_\nu p_\mu=-\dot{S}_{\mu\nu}$. Hence the sum

$$J_{\mu\nu}=L_{\mu\nu}+S_{\mu\nu} \tag{12}$$

is a constant of motion, the total angular momentum, with $S_{\mu\nu}$ being the spin variables. From (5),

$$\ddot{S}_{\mu\nu}=-(\dot{v}_\mu p_\nu-\dot{v}_\nu p_\mu)=4p^\rho(p_\nu S_{\rho\mu}-p_\mu S_{\rho\nu}).$$

The quantity $\Xi_{\mu\rho\nu}=p_\mu S_{\rho\nu}+p_\rho S_{\nu\mu}+p_\nu S_{\mu\rho}$ is a constant of motion, $\dot{\Xi}=0$. Defining $4p_\rho\Xi^{\mu\rho\nu}=-4m^2\Sigma^{\mu\nu}$, we can write

$$\ddot{S}_{\mu\nu}+4m^2S_{\mu\nu}=4m^2\Sigma_{\mu\nu}, \tag{13}$$

where $\Sigma_{\mu\nu}$ is a constant. Consequently we have

$$S_{\mu\nu}=(4m^2)^{-1}(\dot{v}_\mu p_\nu-\dot{v}_\nu p_\mu)+\Sigma_{\mu\nu}, \tag{14}$$

or, in terms of the internal coordinates,

$$S_{\mu\nu}=(Q_\mu p_\nu-Q_\nu p_\mu)+\Sigma_{\mu\nu}. \tag{15}$$

Hence

$$J_{\mu\nu}=X_\mu p_\nu-X_\nu p_\mu+\Sigma_{\mu\nu}=\mathscr{L}_{\mu\nu}+\Sigma_{\mu\nu} \tag{16}$$

with $\dot{\mathscr{L}}_{\mu\nu}=0$, $\dot{\Sigma}_{\mu\nu}=0$.

Thus there are two different decompositions of the total angular momentum, Eqs. (12) and (16). In the second one we have the orbital angular momentum of the center of mass $\mathscr{L}$ plus the orbital angular momentum of the relative motion, which are separately conserved. We can write

$$\Sigma_{\mu\nu}=\mathscr{L}_{\mu\nu}+\Delta_{\mu\nu}=Q_\mu P_\nu-Q_\nu P_\mu+\Delta_{\mu\nu};$$

both $\mathscr{L}_{\mu\nu}$ and $\Delta_{\mu\nu}$ are constants of motions; $\Delta_{\mu\nu}$ (which may be zero) can be interpreted as an *intrinsic* angular momentum in the internal motion.

The symplectic algebra of the representation III [Eqs. (10)] is given by

$$\{X_\mu,X_\nu\}=m^{-1}\Sigma_{\mu\nu}, \quad \{X_\mu,p_\nu\}=g_{\mu\nu},$$
$$\{p_\mu,p_\nu\}=0, \tag{17}$$

$$\{Q_\mu,Q_\nu\}=m^{-2}\Sigma_{\mu\nu}, \quad \{P_\mu,P_\nu\}=4m^2\Sigma_{\mu\nu},$$
$$\{Q_\mu,P_\nu\}=-\tilde{g}_{\mu\nu}\mathscr{H}/m; \quad \tilde{g}_{\mu\nu}=g_{\mu\nu}-m^{-2}p_\mu p_\nu. \tag{18}$$

Note that Eqs. (10), form III, are again of the form $\dot{y}=\{y,\mathscr{H}\}$. From these we derive the following remaining brackets:

$$\{X_\mu,\Sigma_{\alpha\beta}\}=m^{-2}(p_\mu p_\beta Q_\alpha-p_\mu p_\alpha Q_\beta-S_{\alpha\mu}p_\beta+S_{\beta\mu}p_\alpha),$$
$$\{p_\mu,\Sigma_{\alpha\beta}\}=0,$$
$$\{S_{\alpha\beta},\Sigma_{\gamma\delta}\}=\tilde{g}_{\alpha\gamma}S_{\beta\delta}-\tilde{g}_{\beta\gamma}S_{\alpha\delta}-\tilde{g}_{\alpha\delta}S_{\beta\gamma}+\tilde{g}_{\beta\delta}S_{\alpha\gamma}+g_{\alpha\gamma}Q_\beta p_\delta-g_{\beta\gamma}Q_\alpha p_\delta-g_{\alpha\delta}Q_\beta p_\gamma+g_{\beta\delta}Q_\alpha p_\gamma.$$

Note that the internal algebra generated by $(Q_\mu,P_\mu,\Sigma_{\mu\nu},\mathscr{H})$ is closed:

$$\{Q_\mu,\Sigma_{\alpha\beta}\}=\tilde{g}_{\mu\beta}Q_\alpha-\tilde{g}_{\mu\alpha}Q_\beta,$$
$$\{P_\mu,\Sigma_{\alpha\beta}\}=\tilde{g}_{\mu\beta}P_\alpha-\tilde{g}_{\mu\alpha}P_\beta, \tag{19}$$
$$\{\Sigma_{\alpha\beta},\Sigma_{\gamma\delta}\}=\tilde{g}_{\alpha\gamma}\Sigma_{\beta\delta}-\tilde{g}_{\beta\gamma}\Sigma_{\alpha\delta}-\tilde{g}_{\alpha\delta}\Sigma_{\beta\gamma}+\tilde{g}_{\beta\delta}\Sigma_{\alpha\gamma}.$$

Further,

$$\{Q_\mu, X_\nu\} = -m^{-2}(S_{\mu\nu} - \Sigma_{\mu\nu}), \quad \{Q_\mu, p_\nu\} = 0,$$
$$\{P_\mu, X_\nu\} = m^{-1} p_\mu p_\nu + m^{-3} p_\mu p_\nu \mathscr{H}, \quad \{P_\mu, p_\nu\} = 0. \tag{20}$$

The algebras (17)–(20) have their exact counterpart in the proper-time formulation of the quantum Dirac electron.[10] Note, however, that the commutators in (8), and (17)–(20), are independent of the four-dimensional γ matrices; hence they have in quantum theory, in addition to the Dirac representation, more-dimensional representations, which would involve other fermion states, besides the electron and positron.

Our dynamical system characterized by Eqs. (5) has a more natural form in a five-dimensional space-time. We define a five-velocity by

$$v_a = (v_\mu, v_5), \quad v_\mu = \bar{z}\gamma_\mu z,$$
$$v_5 = i\bar{z}\gamma_5 z; \quad a = 0, 1, 2, 3, 5. \tag{21}$$

We omit the index 4 and use 5 instead to avoid confusion. Now v_a is a unit five-vector, although v_μ is not, i.e., $v^a v_a = 1$. The metric is $(+ - - - -)$. The extension of $S^{\mu\nu}$ is

$$S_{5\mu} = -\tfrac{1}{2} a_\mu = -\tfrac{1}{2}\bar{z}\gamma_\mu\gamma_5 z, \quad S_{55} = 0. \tag{22}$$

Further with $p_5 = 0$, $\pi_5 = 0$, $F_{5\mu} = 0$, $F_{55} = 0$, our system in form II [Eqs. (5)] becomes the following (II′):

$$\dot{x}_a = v_a, \quad \dot{v}_a = 4S_{ab}\pi^b, \quad \dot{\pi}_a = eF_{ab}v^b,$$
$$\dot{S}_{ab} = \pi_a v_b - \pi_b v_a, \tag{23}$$

with constraints

$$v_a v^a = 1, \quad S_{ab}v^b = 0 \tag{24}$$

and the constant of motion

$$\mathscr{H} = v^a \pi_a = m. \tag{25}$$

The four first-order equations (23) can be transformed into a single third-order equation in x. First, from (23), $\dot{S}_{ab}v^b = \pi_a - m v_a$, and differentiating this, we have

$$m\dot{v}_a + \ddot{S}_{ab}v^b + \dot{S}_{ab}v^b = eF_{ab}v^b. \tag{26}$$

But, using (23) and (24), we can eliminate $\dot{S}, \ddot{S}$ in terms of S, and obtain finally (IV)

$$m\ddot{x}_a = eF_{ab}\dot{x}^b + S_{ab}\dddot{x}^b. \tag{27}$$

It is interesting that in this form our system is in correspondence with the Lorentz-Dirac equation which describes a spinless particle (i.e., no internal degrees of freedom) but includes radiation reaction. This is in agreement with the one-to-one correspondence between the quantum Dirac equation and the classical Lorentz-Dirac equation found earlier.[11] The Lorentz-Dirac equation ($c = 1$),

$$m\ddot{x}_\mu = eF_{\mu\nu}\dot{x}^\nu + \tfrac{2}{3}e^2[\dddot{x}_\mu + (\ddot{x}^2)\dot{x}_\mu],$$

can be written as

$$m\ddot{x}_\mu = eF_{\mu\nu}\dot{x}^\nu + \tfrac{2}{3}e^2 \mathscr{G}_{\mu\nu}\dddot{x}^\nu, \tag{28}$$

with

$$\mathscr{G}_{\mu\nu} = g_{\mu\nu} - v_\mu v_\nu,$$

because of the identity $(\ddot{x})^2 = -\dot{x}_\nu \dddot{x}^\nu$. The difference between (27) and (28) is that $S_{\mu\nu}$ is antisymmetric, whereas $\mathscr{G}_{\mu\nu}$ is symmetric.

One of us (N.Z.) was the recipient of a fellowship from "Fondazione Angelo Della Riccia."

[1] W. Pauli, Z. Phys. **31**, 765 (1925).

[2] H. A. Kramers, *Quantum Mechanics* (Dover, New York, 1964), Chap. 6.

[3] E. Schrödinger, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl. **24**, 418 (1930).

[4] A. O. Barut and A. J. Bracken, Phys. Rev. D **23**, 2454 (1981).

[5] A. O. Barut and A. J. Bracken, Phys. Rev. D **24**, 3333 (1981).

[6] A. O. Barut, in *Differential Geometric Methods in Mathematical Physics,* edited by H. D. Doebner, Lecture Notes in Mathematics Vol. 905 (Springer, Berlin, 1982), p. 90.

[7] Z. Grossmann and A. Peres, Phys. Rev. **132**, 2346 (1963).

[8] A. Proca, J. Phys. Radium **15**, 5 (1954).

[9] A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Dover, New York, 1980).

[10] A. O. Barut and W. Thacker, to be published.

[11] A. O. Barut, in *Quantum Optics, Relativity and Theory of Measurement,* edited by P. Meystre (Plenum, New York, 1983), p. 155.