Anomalous Diffusion in Intermittent Chaotic Systems

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It is shown that anomalous diffusion (i.e., nonlinear growth of mean square displacements) can be caused by a specifically chaotic mechanism. It depends on deterministic diffusion and intermittency and gives rise to mean square displacements which grow asymptotically like t^ν with $0 < \nu \leqslant 1$ or like $t/\ln t$, depending on universality classes.

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The diffusion processes most frequently found in nature exhibit a linear increase of mean square displacements for large times. However, anomalous cases have shown up where the linear evolution is not obeyed. Such anomalous diffusion has been observed experimentally, e.g., in polymer melts[1] and (via conductivity measurements) for charge transport in amorphous solids[2,3] and in K-hollandite,[4,5] a superionic conductor. Monte Carlo simulations reveal anomalous tracer diffusion in concentrated lattice gases.[6] The physical mechanisms underlying the theoretical work depend essentially upon the existence of either disorder[7] or fractal structure.[8]

In the present Letter we show that anomalous diffusion can also be caused by a very different, specifically chaotic mechanism occurring in completely ordered and even deterministic systems. This is the first observation of asymptotically anomalous diffusion ($t \to \infty$) in chaotic systems. The mechanism depends on the occurrence of intermittent chaos,[9–11] in particular, type-III intermittency as studied by Manneville[9] and Procaccia and Schuster.[10] With increasing nonlinearity (characterized by an exponent z) we find a transition from normal diffusion ($z < 2$) to anomalous diffusion ($z > 2$) where mean square displacements increase asymptotically like $t^{1/(z-1)}$. For $z=2$ they diverge like $t/\ln t$. These results are universal, depending only on the exponent z. We conclude by discussing the relevance of our theory to driven Josephson junctions, where self-generated diffusion has already been found.

In a previous paper of one of the authors[12] it was shown that maps that have discrete translational symmetry can generate a *deterministic* diffusive motion. They describe, e.g., the large-friction regime of driven Josephson junctions (see below) and are given by

$$x_{t+1} = f(x_t), \quad t = 0, 1, 2, \ldots, \tag{1}$$

where x_t is the diffusing variable and f is continuous with the symmetries

$$f(n+x) = n + f(x), \quad n \text{ integer}, \qquad f(-x) = -f(x). \tag{2}$$

As a result of Eq. (2) we can introduce cells of length 1 centered at $x = 0, \pm 1, \pm 2, \ldots$. Specifying the map for $0 \leqslant x \leqslant \frac{1}{2}$ completely defines it on the real axis. In all cases discussed so far the diffusive motion generated by such systems turned out to be asymptotically normal.[12–16] Because of symmetry [Eq. (2)] the map must have fixed points at $x = \pm n/2$. Here we study a case where the fixed points in the centers of the cells become marginally stable (slope = 1). The maps then have the limiting form

$$x_{t+1} = x_t + a x_t^z \quad \text{for} \quad x_t \to +0 \tag{3}$$

($a > 0$, $z > 1$). It is known (from nondiffusive maps)[9,10] that this limiting form causes intermittency and $1/f$ noise. We assume that f is smaller than $\frac{1}{2}$ and has a slope larger than unity in a laminar range $0 < x < x_c$ and maps parts of the transfer range $x_c \leqslant x \leqslant \frac{1}{2}$ to neighboring cells. A smooth probability of injection into a neighborhood of the centers is ensured by some weak requirements.[17] Particular examples for Eqs. (1)–(3), which we consider only for illustrative purposes and for numerical simulations, are

$$f(x)=\begin{cases} x+ax^z, & 0\leqslant x<x_c, \\ \dfrac{3}{2}-\dfrac{|4x-1-2x_c|}{1-2x_c}, & x_c\leqslant x\leqslant \frac{1}{2}, \end{cases} \quad (4)$$

with $z>1$ and $a=(1-2x_c)/2x_c^z$. The diffusion mechanism can now be understood as indicated in Fig. 1 where such an example is illustrated. An orbit injected at x_0 in the laminar range of a unit cell eventually reaches the transfer range and is injected into the neighboring cell. The residence time T in a given cell can be arbitrarily long as x_0 may be arbitrarily close to the center of the cell.

In the following, the distribution of residence times $\Psi(T)$, Eq. (7), is derived in analogy to the distribution of laminar times in nondiffusive intermittency that was calculated by Procaccia and Schuster[10] and similarly by Manneville[9] and Hirsch, Huberman, and Scalapino.[11] With their continuous-time approximations Eq. (3) turns into a differential equation that has the solution

$$x_t=[1/x_0^{z-1}-a(z-1)t]^{-1/(z-1)}. \quad (5)$$

The orbit x_t leaves the cell when $x_t\geqslant\frac{1}{2}$. From Eq. (5) we obtain the residence time T in the cell as a function of the injection point x_0:

$$T(x_0)=(1/x_0^{z-1}-2^{z-1})/a(z-1). \quad (6)$$

The distribution $\Psi(T)$ of residence times is thus related to the (unknown) distribution $P_{\rm in}(x_0)$ of injection points, $\Psi(T)\,dT\propto P_{\rm in}(x_0)\,dx_0$, implying

$$\Psi(T)\propto P_{\rm in}(x_0)\,|dx_0/dT|. \quad (7)$$

This is calculated from Eq. (6) and expansion of $P_{\rm in}(x_0)$ around $x_0=0$. The zeroth order $P_{\rm in}(x_0=0)=\text{const}$ gives the leading order for $T\to\infty$. With proper normalization this leads to

$$\Psi(T)=2a[2^{z-1}+a(z-1)T]^{-z/(z-1)}. \quad (8)$$

Note that this distribution does not possess any finite moments for $z\geqslant 2$. According to Eq. (8), $\Psi(T)$ decays with a power law $T^{-z/(z-1)}$ for $T\to\infty$. The accuracy of this result has been checked numerically (Fig. 2).

One can now apply random-walk theory (continuous-time random walk[18]). Denoting Laplace transforms by $\mathscr{L}\{\Psi(T)\}=\tilde{\psi}(s)$ one obtains for the mean square displacement $\sigma^2(t)=\langle\Delta x^2(t)\rangle$

$$\mathscr{L}\{\sigma^2(t)\}=\tilde{\psi}(s)/[s-s\tilde{\psi}(s)], \quad (9)$$

where the transform $\tilde{\psi}(s)$ of Eq. (8) is

$$\tilde{\psi}(s)=\begin{cases} 2^{1-z}ab(bs)^{\nu}\exp(bs) \\ \qquad\times\Gamma(-\nu,bs), & z\neq 2, \\ 1+(2s/a)\exp(2s/a) \\ \qquad\times\text{Ei}(-2s/a), & z=2. \end{cases} \quad (10)$$

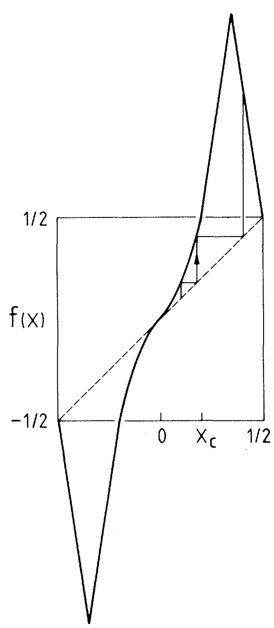

FIG. 1. Example [Eq. (4)] for our classes of models Eqs. (1)–(3). The map is continued along the real axis according to Eq. (2). As indicated by the staircase an orbit starting at $x_0=0.1$ is eventually injected into the neighboring cell $\frac{1}{2}<x<\frac{3}{2}$.

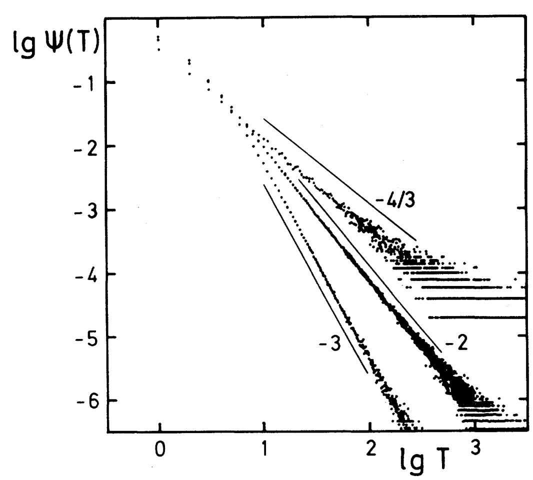

FIG. 2. Distribution of residence times obtained from numerical simulations of Eq. (4) with $z=1.5$, 2, and 4. The lines indicate the asymptotic slopes $-z/(z-1)$ according to Eq. (8).

Here $\nu = 1/(z-1)$ and $b = 2^{z-1}/a(z-1)$. $\Gamma(-\nu, bs)$ is the incomplete gamma function and $\mathrm{Ei}(-2s/a)$ the exponential integral. The long-time behavior of the mean square displacement follows from the small-s behavior of Eq. (9) via Tauberian theorems, in particular Karamata's theorem[19] for $z=2$:

$$\sigma^2(t) \underset{t\to\infty}{\simeq} \begin{cases} (2-z)2^{1-z}t, & 1<z<2, \\ (a/2)t/\ln t, & z=2, \\ \dfrac{a^\nu \sin(\pi\nu)}{2\pi\nu^{z\nu}} t^\nu, & z>2. \end{cases} \qquad (11)$$

The asymptotic time dependence is not influenced by the details of the map but only depends on the universality exponent z.

We have performed numerical simulations illustrating the approach to the asymptotic regime (Fig. 3). The mean square displacements $\sigma^2(t)$ have been computed for an ensemble of 1000 orbits generated by Eqs. (1) and (4) with $x_c = 0.2$. $\sigma^2(t)$ grows linearly for $z=1.5$ as predicted. For $z=3$ and $z=4$ deviations from normal diffusion become noticeable around $t=10^3$. For later times the asymptotic exponents $\frac{1}{2}$ and $\frac{1}{3}$ are approached steadily but slowly. This slow convergence is due to the very infrequent occurrence of intermittent bursts[9] for $z>2$. For $z=2$ the logarithmic deviations show up only weakly in the upper part of Fig. 3 but can be detected in the lower diagram. In this plot the $t/\ln t$ trend is reflected in the linear increase above $t=10^3$, whereas the curve should remain constant for normal diffusion ($z=1.5$).

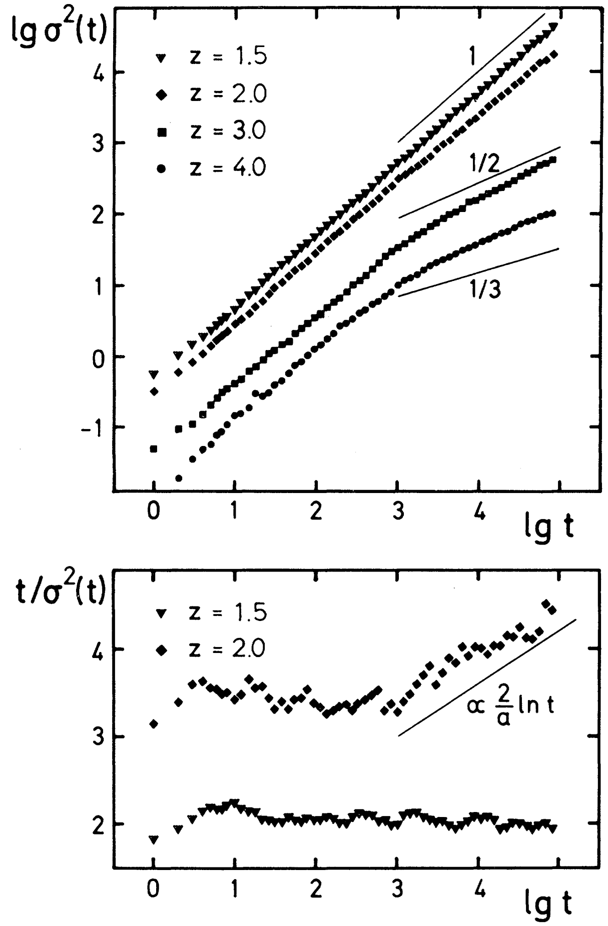

FIG. 3. Mean square displacements of an ensemble of 1000 orbits vs time. The log-log plot (top) shows the deviations from normal diffusion (slope 1) for $z>2$. The logarithmic correction for $z=2$ is revealed more clearly by the t/σ^2 vs $\log t$ plot (bottom). The lines indicate the asymptotic slopes predicted by Eq. (11).

The anomalous diffusion found here is due to the existence of a statistical distribution $\Psi(T)$ of residence times with a long-time tail. In previous theories[7] such distributions had to be assumed, e.g., as being due to different residence times at different positions in a disordered medium. In the present chaotic system we have found a new and quite different mechanism for anomalous diffusion. The distribution is brought forth very naturally by a mechanism that depends on intermittency and deterministic diffusion and does not require any spatial disorder.

We finally discuss physical applications of this theory. Deterministic diffusion has already been observed in driven Josephson junctions.[20] Their asymptotic dynamics in the strongly dissipative case is well described by one-dimensional maps like Eq. (1). This is because dissipation leads to a contraction of volume elements in phase space. Thus one often finds attractors with dimensions close to two, which can be reduced (in good approximation) to one-dimensional maps by Poincaré section. Recently such a map has been determined for Josephson junctions.[14] It even exhibits intermittent diffusion although of a different type. The requirements for finding a map like Eq. (3) are weak: Reflection symmetry ($x \to -x$) implies vanishing of the quadratic expansion term around a fixed point ($x=0$). When the fixed point reaches marginal stability the expansion is given by Eq. (3) with $z=3$, which according to our theory entails anomalous diffusion with $\nu = \frac{1}{2}$. In view of their large parameter space (four dimensional), we believe that this situation will show up in Josephson junctions.[21] As an analogy we quote that a (nondiffusive) map similar to Eq. (3) has recently been measured in a Rayleigh-Bénard experiment.[22]

After completion of this paper we became aware of an additional mechanism for anomalous diffusion in *nondeterministic* systems.[23]

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