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## Supersymmetry and the Bistable Fokker-Planck Equation

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The approach to equilibrium in a bistable system is governed by a small eigenvalue which appears to be difficult to compute. We show that supersymmetry makes possible an easy evaluation of this eigenvalue.

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Diffusion of a particle in a one-dimensional, bistable potential provides a useful model to understand the role of thermal fluctuations in driving a system towards equilibrium.<sup>1</sup> The particle moves in a highly viscous medium so that its inertial reaction (acceleration) can be neglected. It is acted on by a force which is the gradient of the potential plus a random (thermal) force described by Gaussian-correlated, white noise. Associated with the particle position is a time-dependent probability density  $P(x,t)$  which obeys the Fokker-Planck equation. Although there exists a unique, time-independent equilibrium distribution  $P_0(x)$ , the manner in which this distribution is reached may be delicate. With a bistable potential  $U(x)$  as shown in Fig. 1, the particle can become trapped in one of the two potential wells, and the approach to equilibrium will take place very slowly at low temperatures.

In this Letter we exploit the correspondence<sup>2</sup> between the one-dimensional Fokker-Planck equation with an arbitrary potential and supersymmetric quantum mechanics.<sup>3</sup> This correspondence allows a simple calculation of the small eigenvalue of the Fokker-Planck equation, the eigenvalue that controls the rate at which equilibrium is approached.

We write the Fokker-Planck equation as

$$\frac{\partial}{\partial t} P(x,t) = \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial x} + \beta U'(x) \right] P(x,t), \quad (1)$$

where  $U' = dU/dx$  and  $1/\beta$  is proportional to the temperature. Any initial distribution  $P(x,t)$  will relax at very large times to the time-independent, equilibrium solution

$$P_0(x) = N_0 \exp\{-\beta U(x)\}, \quad (2)$$

where  $N_0$  is a normalization constant. To discuss intermediate times, we write

$$P(x,t) = \psi(x,t) \exp\{-\frac{1}{2}\beta U(x)\}, \quad (3)$$

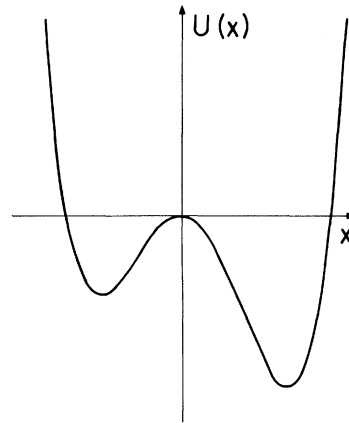


FIG. 1. Shape of the bistable potential in the Fokker-Planck equation.

for then  $\psi(x, t)$  satisfies a Schrödinger equation in imaginary time

$$\partial\psi/\partial t = -H\psi \quad (4)$$

with a Hermitian, positive semidefinite Hamiltonian operator

$$H = A^\dagger A, \quad (5)$$

where

$$A = \frac{1}{2}\beta U' + \partial/\partial x, \quad (6a)$$

and

$$A^\dagger = \frac{1}{2}\beta U' - \partial/\partial x. \quad (6b)$$

By introduction of a complete set of eigenvectors  $\{\psi_n\}$ ,

$$H\psi_n = \lambda_n \psi_n, \quad (7)$$

the general solution to Eq. (4) may be written as

$$\psi(x, t) = \sum_n C_n \exp[-\lambda_n t] \psi_n(x). \quad (8)$$

The coefficients  $C_n$  are determined by the initial distribution. It is easy to see that  $\psi_0 = \exp\{-\frac{1}{2}\beta U\}$  is the unique<sup>4</sup> eigenfunction with  $\lambda_0 = 0$ . However, in the low-temperature limit the first nonvanishing eigenvalue  $\lambda_1$  is exceedingly small since the system must take a long time to reach equilibrium. It would appear that a rather difficult (tunneling) calculation is required for the evaluation of  $\lambda_1$ . We turn now to show that supersymmetry yields an easy determination of  $\lambda_1$ .

To put Eq. (7) in a supersymmetric form, one extends the wave function  $\psi$  to a two-component column  $\Psi$  and defines

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad (9a)$$

$$Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}. \quad (9b)$$

These are nilpotent, "Fermionic" operators,  $Q^2 = 0 = Q^{\dagger 2}$ . The supersymmetric Hamiltonian  $H_{ss}$  which acts on  $\Psi$  is given by

$$H_{ss} = Q^\dagger Q + Q Q^\dagger = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}. \quad (10)$$

Here  $H_+ = A^\dagger A = H$  is the original Hamiltonian associated with the Fokker-Planck equation while  $H_- = A A^\dagger$ . Thus the upper, "bosonic" component of  $\Psi$  contains the previous eigenvectors  $\psi_n$ . Since  $Q$  is nilpotent,  $Q$  commutes with  $H_{ss}$ . Hence if  $\Psi_n$  is an eigenvector so is  $Q\Psi_n$ . Thus there are two

towers of states with degenerate eigenvalues. The application of  $Q$  to a bosonic  $\Psi$  with only an upper component produces a fermionic  $\Psi$  with only a lower component. The only exception is the ground state  $\Psi_0$ . Since  $A\Psi_0 = 0$ , it has no fermionic counterpart. The Hamiltonians  $H_\pm$  which operate in the two sectors,

$$H_\pm = d^2/dx^2 + V_\pm, \quad (11)$$

define bosonic (+) and fermionic potentials (−):

$$V_\pm = [\frac{1}{2}\beta U']^2 \mp \frac{1}{2}\beta U''. \quad (12)$$

With a bistable potential  $U(x)$  [Fig. 1], the bosonic potential  $V_+$  has two wells, but the fermionic potential  $V_-$  has essentially only a single well. The eigenvalue and potential structure which we have just discussed is illustrated in Fig. 2.

Now, to calculate the excited ( $\lambda_n > 0$ ) levels of the original Fokker-Planck equation, we may instead work on the fermionic ladder. This is a system whose spectrum is much easier to obtain than is that of the original Fokker-Planck Hamiltonian. In particular, it is easy to obtain an upper bound for the small eigenvalue  $\lambda_1$  since it appears in the ground state of the fermionic ladder. We may use the variational principle with the trial wave function having only the lower component

$$\psi_-(x) = \exp\{\frac{1}{2}\beta\Phi(x)\}. \quad (13)$$

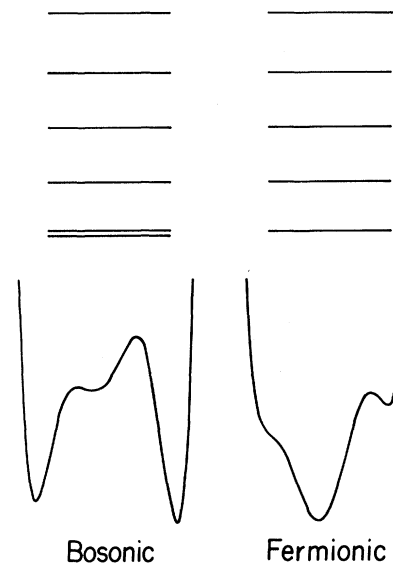


FIG. 2. The two degenerate towers of eigenvalues which arise from supersymmetry. The equivalent Schrödinger equation bosonic,  $V_+$ , and fermionic,  $V_-$ , potentials are sketched beneath their eigenvalue towers.

The wave function  $\exp\{\frac{1}{2}\beta U(x)\}$  is annihilated by  $A^\dagger$  and thus would appear to be an eigenfunction of  $H_-$  with eigenvalue zero. This function, however, is not normalizable. We obtain a normalizable function which is continuous in both value and slope by fitting  $\exp\{\frac{1}{2}\beta U(x)\}$  to decreasing exponentials. Explicitly, we take

$$\Phi(x) = \begin{cases} U(-c) + U'(-c)(x+c), & x \leq -c, \\ U(x), & -c < x < d, \\ U(d) + U'(d)(x-d), & x \geq d, \end{cases} \quad (14)$$

where  $c$  and  $d$  are variational parameters. We use the force

$$U'(x) = x(x+a)(x-b). \quad (15)$$

In the low-temperature limit of interest,  $\beta \rightarrow \infty$ , the variational principle provides the rigorous upper bound

$$\lambda_1 \leq \frac{\beta ab}{6(6\pi)^{1/2}} \left[ \left( \frac{a+b}{b} \right)^{1/2} \exp\left\{ \frac{3}{2} - \frac{\beta a^3}{12}(a+2b) \right\} + \left( \frac{a+b}{a} \right)^{1/2} \exp\left\{ \frac{3}{2} - \frac{\beta b^3}{12}(b+2a) \right\} \right]. \quad (16)$$

Note that the eigenvalue is exponentially small as  $\beta \rightarrow \infty$ . On the other hand, near the bottom of the fermionic well ( $x \sim 0$ ),

$$V_-(x) = (\frac{1}{2}\beta abx)^2 - \frac{1}{2}\beta ab. \quad (17)$$

Thus the low-lying levels are approximately described by the harmonic oscillator eigenvalues,

$$\lambda_n = E_{n-1}^{\text{HO}} - \frac{1}{2}\beta ab = (n-1)\beta ab. \quad (18)$$

[This gives  $\lambda_1 = 0$  which should be replaced by the more accurate Eq. (16).] The higher eigenvalues  $\lambda_n$  with  $n \geq 2$  increase linearly with  $\beta$ . Hence, at low temperatures, only the first two terms in the sum of Eq. (8) are important in the long-time limit.

Our method can be used in other cases as well. A particularly straightforward example is the critical case<sup>5</sup> where  $a = b = 0$  and  $U' = x^3$ . We now set  $\beta = 1$ . The bosonic potential  $V_+ = \frac{1}{4}x^6 - \frac{3}{2}x^2$  is a double well whose eigenvalues are difficult to determine. The fermionic potential  $V_- = \frac{1}{4}x^6 + \frac{3}{2}x^2$  corresponds to a single-well anharmonic oscillator whose low-lying eigenvalues may be readily computed with the variational principle. With the trial wave functions  $\psi_{-1} = \exp\{-\frac{1}{2}\alpha_1 x^2\}$  and  $\psi_{-2} = x \exp\{-\frac{1}{2}\alpha_2 x^2\}$  it is a simple matter to find that the first two nonvanishing eigenvalues have the bounds  $\lambda_1 \leq 1.38$ ,  $\lambda_2 \leq 4.51$ . These are to be compared with the results obtained directly from the bosonic Hamiltonian<sup>5</sup>: A computer calculation in-

volving a  $100 \times 100$  matrix produces  $\lambda_1 \approx 1.37$ ,  $\lambda_2 \approx 4.45$ ; a variation method gives  $\lambda_1 \approx 1.48$ ,  $\lambda_2 \approx 4.97$ ; a WKB computation yields  $\lambda_1 \approx 1.66$ ,  $\lambda_2 \approx 4.70$ .

In summary, the latent supersymmetry of the one-dimensional Fokker-Planck equation can be exploited to obtain an accurate estimate of the very small eigenvalue in the bistable problem. The supersymmetry can also be used to replace a difficult double-well problem with an equivalent but much simpler single-well problem.

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<sup>1</sup>See, for example, N. G. van Kampen, J. Stat. Phys. **17**, 71 (1977), and references therein.

<sup>2</sup>C. M. Bender, F. Cooper, and B. Freedman, Nucl. Phys. **B219**, 61 (1983). The Appendix to this paper also contains a derivation of the Fokker-Planck equation from the Langevin equation describing the particle's motion in the random force.

<sup>3</sup>E. Witten, Nucl. Phys. **B185**, 513 (1981).

<sup>4</sup> $(\psi, H\psi) = 0$  implies that  $A\psi = 0$  and hence  $\psi = \psi_0$ .

<sup>5</sup>H. Dekker and N. G. van Kampen, Phys. Lett. **73A**, 374 (1979).