# Monte Carlo Random-Walk Experiments as a Test of Chaotic Orbits of Maps of the Interval

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We have performed Monte Carlo random-walk experiments on a one-dimensional periodic lattice with a trapping site using the logistic map as a generator of pseudorandom numbers. Comparison with analytical results indicates that, when it has sensitive dependence to the initial conditions, this map provides a true pseudorandom generator.

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Iterations of continuous maps of an interval onto itself provide the simplest examples of models for dynamical systems.[1] In spite of their structural simplicity, such models exhibit a variety of behaviors, including limit points, limit cycles, and chaotic attractors. In the present work, we focus our attention to those regimes where the behavior (although fully deterministic) appears as *chaotic,* i.e., shows profound similarities to the sample function of a random process. For a given initial condition, the iteration of a continuous map generates an orbit which yields a sequence of pseudorandom numbers. The frequency of realization of a particular number within a given interval defines a probability density which must be independent of the initial condition in order that such a generator qualify as a *random generator.*[2] In a mathematical sense, this amounts to imposing the existence of an invariant measure, absolutely continuous with respect to Lebesgue measure.

However, the existence of absolutely continuous measures has been proved only in a limited number of cases. No general theorem is available but a conjecture by Ruelle for general dynamical systems states that if a system possesses sensitive dependence to initial conditions (i.e., a system with positive Liapunov characteristic exponents), then the invariant measure(s), when conditioned onto the unstable manifold(s), should be absolutely continuous.[3]

Now, restriction to maps of the interval makes the case easier because when there is an unstable direction, then there is no space for a stable direction. When the map is everywhere expanding, the Liapunov characteristic exponent is clearly positive, and in this case the existence of an absolutely continuous invariant measure was shown.[4,5] A map of the interval with a critical point is of course not uniformly expanding; however, it was proved that there is an absolutely continuous invariant measure when the critical points have orbits which eventually land on unstable fixed points.[6,7] This result holds for maps whose critical points have orbits that do not come close to the critical point.[8,9] Very recently, it was shown that an absolutely continuous invariant measure exists when the Liapunov exponent is positive and the inverse of the map is contracting.[10]

These rigorous results appear to converge and lend support to Ruelle's conjecture. However, a dissonant claim[11] was recently presented on the basis of a numerical study of the logistic map of the interval [0,1]:

$$x_{n+1} = f(x_n) = Rx_n(1-x_n). \tag{1}$$

Kozak, Musho, and Hatlee considered this map as a pseudorandom number generator and they used Monte Carlo simulation to calculate the average walk length for trapping on a periodic one-dimensional lattice with a centrosymmetric trap. They arrived at the conclusion that comparison with exact results suggests that the only truly chaotic sequence is the one for $R=4$.[11]

In the present work we reconsider such a numerical simulation in order to clear up the underlying experimental bias where the apparent discrepancy between Kozak, Musho, and Hatlee's conclusion and Ruelle's conjecture originates. When comparing numerical data with exact theoretical results, one must be careful about the deterministic character of the sequences of pseudorandom numbers generated by the iteration of the logistic map. In particular one must eliminate the short-range correlations inherent to such an iterative procedure.

This can be performed in two complementary ways: either by introducing such correlation effects in a reformulation of the theory, or by numerically removing these correlations by taking some high iterate of the logistic map as the random number generator. Using both methods we obtain results which give strong experimental support to Ruelle's conjecture.

The use of the nonlinear map (1) as a random number generator for a one-dimensional random-walk experiment requires characterization of the interval explored by the successive iterates of (1): For given $R$, this interval is $[\frac{1}{4}R^2(1-\frac{1}{4}R),\frac{1}{4}R]$. Then by defining a cutoff separating this interval into two subintervals, one makes the random walker step either to the right or to the left depending on which subinterval the iteration lies in. If there exists an absolutely continuous invariant measure $\rho$ defined on the orbit (sequence of numbers), then the problem is well defined in the sense that the walker has on the average a probability $p$ for moving to the right and $1-p$ for moving to the left; here $p$ is defined as the $\rho$ measure of the right subinterval.

In previous work,[11] the symmetric case $p=1-p=\frac{1}{2}$ was investigated; in particular Montroll[12] proved that the expected walk length for a walker to be trapped starting from the site $s$ is given by

$$\langle n(s)\rangle_{\text{theor}}=s(N-s) \tag{2}$$

where $N$ is the lattice period. Note that the symmetric probability condition imposes a rather drastic, but unnecessary [and possibly misleading (see below)] constraint, which can be relaxed. Montroll's result then generalizes to[13]

$$\langle n(s)\rangle_{\text{theor}}=\frac{1}{1-2p}\left(-s+N\frac{1-[p/(1-p)]^s}{1-[p/(1-p)]^N}\right). \tag{3}$$

In the numerical experiment, the expected walk length is evaluated as follows: Simulations are performed for $\mathcal{N}$ ($=2000$ to $800\,000$) walkers starting from each of the $N-1$ nontrapping lattice sites. The numerical average $\langle n(s)\rangle_{\text{expt}}$ is obtained by calculating $n(s)$ for each of these sites and averaging over the $\mathcal{N}$ walkers. In the absence of any bias in the sequence of pseudorandom numbers, the numerical estimate $\langle n(s)\rangle_{\text{expt}}$ should be close to $\langle n(s)\rangle_{\text{theor}}$ within statistical error; the statistical error decreases with increasing $\mathcal{N}$ as $1/\mathcal{N}^{1/2}$ for $\mathcal{N}$ large, according to the central-limit theorem. Note that in general the statistical error is not only a function of $\mathcal{N}$ but also of $s$, $p$, and $N$. Our calculation of the statistical error is in excellent agreement with the numerical value (see below).

For $R=4$, the map (1) possesses an absolutely continuous invariant measure with respect to Lebesgue measure with density $\rho(x)=(1/\pi)\times[x(1-x)]^{1/2}$.[14] Numerical experiments carried out with symmetric cutoff $x_c=\frac{1}{2}$ $(p=\frac{1}{2})$ yield results similar to those reported in Ref. 11; see Table I, 1.1. Experiment and theory are in agree-

TABLE I. Average length for trapping from individual site, $\langle n(s)\rangle$, on an $N=13$ periodic lattice. We restrict our presentation to two representative starting sites. Complete results will be given in Ref. 13; $\langle n(s)\rangle_{\text{theor}}$ is obtained from Eq. (3).

| | $s$ | $\langle n(s)\rangle_{\text{expt},f}$ | $\langle n(s)\rangle_{\text{expt},f^2}$ | $\langle n(s)\rangle_{\text{expt},f^n}$ | $\langle n(s)\rangle_{\text{theor}}$ |
|---|---|---|---|---|---|
| 1.1: $R=4$, | 8 | 39.93 | | | 40.00 |
| $p=\frac{1}{2}$, $x_c=\frac{1}{2}$ | 9 | 36.29 | | | 36.00 |
| 1.2: $R=4$, | 8 | 20.98 | 23.46 | 22.75[a] | 22.76 |
| $p=\frac{2}{3}$, $x_c=\frac{1}{4}$ | 9 | 22.06 | 25.74 | 24.75[a] | 24.54 |
| 1.3: $R=3.8$, | 8 | 103.6 | | 38.91[b] | 40.00 |
| $p=\frac{1}{2}$, $x_c=0.6902$ | 9 | 91.51 | | 34.97[b] | 36.00 |
| 1.4: $R=3.8275$, | 8 | 16.64 | | 17.62[b] | 17.52 |
| $p=0.7254$, $x_c=0.4$ | 9 | 18.75 | | 19.58[b] | 19.37 |
| 1.5: $R=3.62$, | 2 | | 26.39 | 21.36[b] | 21.12 |
| $p=0.2919$, $x_c=\frac{1}{2}$ | 3 | | 24.09 | 22.12[b] | 21.84 |

[a] $n=10$. [b] $n=20$.

ment within statistical error; the mean standard deviation is of the order of 1% for $\mathscr{N}=20\,000$. However, considering the monotonic behavior of the map (1) on each subinterval, one could expect a systematic discrepancy due to the existence of short-range correlations. By direct numerical computation, we find that two-point correlation functions (with the condition that the first point belongs to one of the two subintervals) show exponential decay with a correlation length equal to the Liapunov characteristic exponent, $\lambda=\ln 2$. Although such correlation effects are always experimentally present, they are buried in the statistical error, because of their low amplitude. In order to make them observable in the random-walk experiment, fairly prohibitive statistics ($\mathscr{N}\geqslant 10^7$) would be required.

It is important to realize that, in order to test the existence of an absolutely continuous invariant measure, investigations cannot be limited to one particular cutoff ($x_c=\frac{1}{2}$), and must be extended to values of $p$ between 0 and 1. Now, for arbitrary cutoff values, a new correlation effect enters the problem; indeed not only must successive steps be independent but each step has to be weighted with the same probability, $p$ or $1-p$, when the walker moves to the right or to the left. As soon as one of the boundaries of either subinterval, $[\frac{1}{4}R^2(1-\frac{1}{4}R),x_c]$ or $[x_c,\frac{1}{4}R]$, is not mapped onto one of the boundaries of the whole invariant subinterval $[\frac{1}{4}R^2(1-\frac{1}{4}R),\frac{1}{4}R]$, $p$ becomes a conditional probability which depends on the preceding step. This non-Markovian effect can lead to a strong discrepancy between theory and experiment as illustrated in Table I, 1.2. In this case, failure to get a fixed $p$ for each step arises from partial overlap of the invariant interval with the iterate of one of the two subintervals. Further experiments performed with a higher iterate $f^n$ of the logistic map lead to considerably better agreement with theory. Numerical computation of the probability $p$ for each step indeed shows convergence to the mean value of $p$ as $n$ is increased. Then each subinterval is mapped several times onto the invariant interval with the result that the relative importance of the mismatch diminishes with the number of mappings.

Whatever the cutoff value $x_c$, one can always compensate numerically for the ill definition of $p$ at each step by using a sufficiently high iterate of $f$. However, for $R=4$ and particular cutoff values —that is, the $n$th inverse iterates of the critical point $\frac{1}{2}$ of the logistic map for all $n$—this ill definition of $p$ strictly disappears when one considers $f^{n+1}$ as the random number generator, as illustrated in Table II for $x_c=\inf f^{-1}(\frac{1}{2})$. For this simple case we obtain, as expected, good numerical results when using $f^2$, which reflects a two-step correlation effect. We have also developed a theoretical analysis for this non-Markovian behavior[13] which yields expected walk lengths in agreement with our numerical simulations performed with $f$ (see Table II).

TABLE II. Average walk length for trapping from individual site, $\langle n(s)\rangle$, on an $N=13$ periodic lattice for $R=4$, $p=\frac{3}{4}$ ($x_c=\frac{1}{2}-1/2\sqrt{2}$) and with use of $f$ and $f^2$, respectively. $\langle n(s)\rangle_{\text{theor}}$ is derived from Eq. (3). Exact results obtained for a non-Markovian two-step effect random walk (Ref. 13) are shown in the second column and compare well with experimental results when $f$ is used.

| $s$ | $\langle n(s)\rangle_{\text{theor}}$[a] | $\langle n(s)\rangle_{\text{expt};\,f}$ | $\langle n(s)\rangle_{\text{expt};\,f^2}$ | $\langle n(s)\rangle_{\text{theor}}$ |
|---|---|---|---|---|
| 1 | 1.998 | 1.960 | 1.992 | 2.000 |
| 2 | 3.996 | 3.997 | 3.983 | 4.000 |
| 3 | 5.992 | 5.942 | 6.010 | 6.000 |
| 4 | 7.981 | 7.925 | 7.957 | 7.999 |
| 5 | 9.958 | 9.863 | 10.00 | 9.996 |
| 6 | 11.91 | 11.66 | 11.98 | 11.99 |
| 7 | 13.79 | 13.50 | 13.94 | 13.96 |
| 8 | 15.53 | 15.12 | 15.84 | 15.89 |
| 9 | 16.96 | 16.28 | 17.63 | 17.68 |
| 10 | 17.67 | 16.88 | 19.00 | 19.04 |
| 11 | 16.80 | 16.20 | 19.19 | 19.11 |
| 12 | 12.37 | 12.40 | 15.39 | 15.33 |

[a]Ref. 13.

We now investigate the general case $R \neq 4$. For the set of Misiurewicz's[8] $R$ values, one can easily convince oneself that the problem simply reduces to the case $R=4$, when $f^k$ instead of $f$ is used as a random generator. These $R$ values correspond to the different stages of the reverse cascade[1] for which the $2^{k+1}$-band chaotic attractor merges to $2^k$ bands. Indeed in each of the $2^k$ bands, $f^k$ is surjective as is $f$ on [0,1] for $R=4$.

We first concentrate on values of $R$ in the last step of the reverse cascade, $R = 3.678\,57\ldots$ to $R=4$. For $R=3.8$, and a bisecting cutoff such that $p=\frac{1}{2}$, we obtain the same poor results as those presented in Ref. 11 when $f$ is used. However, with $f^{20}$ considerable improvement is obtained as shown in Table I, 1.3. A numerical estimate of the short-range correlation effects shows unambiguously that the ill definition of $p$ at each step is responsible for the misleading claim of Kozak, Musho, and Hatlee. Our conclusion is corroborated by analysis with different cutoff values, for which the correlation effects are less dramatic[15] and easier to handle with lower iterates of the logistic map.[13]

The results for $R=3.8275$ are presented in Table I, 1.4. For this value of the parameter, the logistic map shows intermittent behavior preceding the occurrence of a stable period-three cycle. This characteristic short-range order in chaotic dynamics is reflected in the random-walk experiment by the necessary use of rather high iterates of $f$. The bias increases drastically when $R$ approaches the bifurcation value $R = 1+\sqrt{8} = 3.828\,427\ldots$. Avoiding this intermittent regularity could be attempted by selecting those points in the dynamics which are not in the resonant channels; but even for such points, memory of the periodicity persists.

We have also extended our numerical study to the different stages of the reverse cascade, and we obtain similar results when the nonconnexity of the chaotic attractor is taken into account. In particular, the disconnected structure of the attractor mirrors into a back and forth walk, for $f$ in the symmetric case $p=\frac{1}{2}$. For a $2^{k-1}$-band attractor, the general procedure requires at least the $k$th iterate of $f$. This restricts the analysis to one of the bands of the attractor which is invariant under $f^k$. Next, by choosing an arbitrary cutoff in this band, we face again a situation similar to that encountered in the last stage of the reverse cascade discussed above. In Table I, 1.5, we present experimental results for $R=3.62$ in the two-band chaotic region. For $f^{20}$ as the random number generator, good agreement is obtained between theory and experiment.

Many additional numerical experiments were performed for other arbitrary values of $R$.[13] They all confirm the results presented above: When the Liapunov characteristic exponent is computed to be positive, the logistic map can be used as a pseudorandom number generator provided that its deterministic nature is taken into account. Therefore we may conclude that the present work lends strong experimental support to Ruelle's conjecture.

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[1]P. Collet and J. P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems* (Birkhäuser, Basel, 1980).

[2]D. E. Knuth, *The Art of Computer Programming* (Addison-Wesley, Reading, Mass., 1969), Vol. 2, pp. 1–160.

[3]D. Ruelle, in *Bifurcation Theory and its Applications in Scientific Disciplines,* edited by O. Gurel and O. E. Rossler, Annals of the New York Academy of Sciences Vol. 316 (New York Academy of Sciences, New York, 1979).

[4]A. Lasota and J. A. Yorke, Trans. Am. Math. Soc. **186**, 481 (1973).

[5]P. Walters, Trans. Am. Math. Soc. **236**, 121 (1975).

[6]R. Bowen, Commun. Math. Phys. **69**, 1 (1979).

[7]D. Ruelle, Commun. Math. Phys. **55**, 47 (1977).

[8]M. Misiurewicz, Publ. Math. IHES **53** (1981).

[9]W. Szlenk, Institut des Hautes Etudes Scientifiques Reports No. M/80/30 and No. M/80/41, 1980 (to be published).

[10]P. Collet and J. P. Eckmann, Ergod. Th. Dynam. Syst. **3**, 13 (1983).

[11]J. J. Kozak, M. K. Musho, and M. D. Hatlee, Phys. Rev. Lett. **49**, 1801 (1982).

[12]E. W. Montroll, Proc. Symp. Appl. Math. Am. Math. Soc. **16**, 193 (1964); E. W. Montroll and G. W. Weiss, J. Math. Phys. **6**, 167 (1965); E. W. Montroll, J. Math. Phys. **10**, 753 (1969).

[13]D. Sornette and A. Arneodo, to be published.

[14]S. M. Ulam and J. Von Neumann, Bull. Am. Math. Soc. **53**, 1120 (1947).

[15]Cutoff values such that $p \sim \frac{1}{2}$ lead to short-range oscillating behavior reminiscent of the systematic periodicity 2 superimposed on chaotic behavior as it occurs in the previous stage of the reverse cascade.