Exotic Structure in the σ Model

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(Received 24 October 1983)

In the Gell-Mann–Levy σ model, although the soliton by itself is unstable to collapse, the author finds a stable self-consistent solution of a valence (single-particle) nucleon *bound* state in a soliton (hedgehog) configuration of the meson fields. This solution has baryon number 2 and in the neglect of quantum fluctuations its energy is found to be less than $m_N = g\langle\sigma\rangle$ $(= gf_\pi)$, the mass of the usual plane-wave nucleon.

PACS numbers: 11.10.Lm, 11.30.Qc, 11.30.Rd

The σ model[1] is a realization of pion-nucleon physics which reproduces the usual current-algebra results. Chiral symmetry is spontaneously broken by the σ field acquiring a vacuum expectation value (VEV) which generates a mass for the nucleon through the Yukawa coupling. In the lowest order, treating the meson fields as classical (VEV) and the nucleon at the single-particle level, we find a plane-wave spectrum for the nucleon with a mass given by $g\langle\sigma\rangle$, where $\langle\sigma\rangle = f_\pi = 93$ MeV, the pion decay constant, and $\langle\underline{\pi}\rangle = 0$ in accordance with the partial conservation of axial-vector current. In these considerations, $\langle\sigma\rangle$, is, of course, a constant over all space.

Here I examine the possibility of a nucleon bound (localized) state arising from a vacuum expectation value for the meson fields which is space dependent in the interior, but clearly finite energy of these states requires that asymptotically $\langle\sigma\rangle = f_\pi$ and $\langle\underline{\pi}\rangle = 0$ as in the usual case. The space variation of $\langle\sigma\rangle$ and $\langle\underline{\pi}\rangle$ in the interior is expected to act as a potential to trap the nucleon.

Such configurations of the σ and π fields can carry nontrivial topology. This was observed by Skyrme[2] years ago. Skyrme introduced an extra term in the pure nonlinear ($\sigma^2 + \underline{\pi}^2 = f_\pi^2$ over all space) meson Lagrangian and found a stable soliton solution in the absence of nucleons. Further, he associated the topological charge of this configuration with baryon number. This has been the subject of much interest lately in the light of a paper by Goldstone and Wilczek[3] who explicitly showed that the topological charge can be identified with baryon number. Balachandran *et al.*[4] have constructed a stable soliton in analogy with Skyrme and shown that it carries topological baryon number 1. In addition they have constructed a bound state of a nucleon in the soliton background with even baryon number and exotic quantum numbers (spin, etc.).

I present here a solution in which the additional term, L_1, introduced by Skyrme[2] to stabilize the soliton in the absence of fermions is not present. Thus, the soliton by itself cannot exist. On the other hand, a self-consistent solution[5] of the coupled *nucleon* and solitonic *boson* fields is developed which is stable and of the same topology as the soliton. This localized (bound-state) solution has baryon number 2 but a very different energy from that of Ref. 4. In particular, it is found to have an energy lower than the plane-wave single nucleon $(B=1)$ $M_N = g\langle\sigma\rangle$. The consequence is extraordinary—an extended topological object of baryon number twice the ordinary nucleon but energy less than its mass is present at the semiclassical level in the σ model.

The starting point is the chiral σ-model Hamiltonian for classical time-independent pion and sigma fields:

$$H = \int d^3r\, \psi^\dagger\{\vec{\alpha}\cdot\vec{\mathrm{p}} + g\beta[\sigma + i\gamma_5(\underline{\pi}\cdot\underline{\tau})]\}\psi + \int d^3r \left[\left\{\left(\frac{\vec{\nabla}\sigma}{2}\right)^2 + \left(\frac{\vec{\nabla}\underline{\pi}}{2}\right)^2\right\} + \lambda(\sigma^2 + \underline{\pi}^2 - \sigma_0^2)^2 - f_\pi m_\pi^2 \sigma\right]. \quad (1)$$

The last term is the explicit symmetry-breaking term, in accordance with the partial conservation of axial-vector current, which vanishes in the Goldstone boson limit of the pion $(m_\pi = 0)$. We work in the limit of no quantum fluctuations, i.e., classical $\underline{\pi}$ and σ fields and single-particle Dirac nucleon fields.

Minimization of the *meson* part of the energy in the absence of the nucleon yields $\langle\sigma\rangle = f_\pi$, $\langle\underline{\pi}\rangle = 0$, and $\sigma_0 = f_\pi^2 - m_\pi^2/\lambda$. This is to indicate that in the presence of even an infinitesimal symmetry-breaking term, the absolute minima is as above, $\langle\sigma\rangle \simeq \sigma_0 = +f_\pi$, and not $\langle\sigma\rangle = -\sigma_0$. I shall, from this point onwards, adopt the $m_\pi = 0$ limit, keeping $\sigma = +f_\pi$. The solution with a localized nucleon source must then satisfy the boundary condition at spatial infinity $\sigma(r\rightarrow\infty) = +f_\pi$.

The energy of the nucleon bound system in the presence of the space-dependent $\underline{\pi}$ and σ fields is obtained by first solving the Dirac equation for the nucleon valence orbital:

$$\{\vec{\alpha}\cdot\vec{p} + g\beta[\sigma + i\gamma_5(\underline{\pi}\cdot\underline{\tau})]\}|n\rangle = e_n|n\rangle, \tag{2}$$

where n designates the eigenvalue. Let e_0 be the ground state or lowest eigenvalue. The total ground-state energy, E, is then given by the addition of the meson part:

$$E = e_0 + \int d^3r\,[\tfrac{1}{2}(\vec{\nabla}\sigma)^2 + \tfrac{1}{2}(\vec{\nabla}\underline{\pi})^2 + \lambda(\underline{\pi}^2 + \sigma^2 - f_\pi^2)^2]. \tag{3}$$

The solution corresponds to E being stationary under independent variations of the σ and $\underline{\pi}$ fields given the line boundary condition $\langle\pi\rangle_{r\rightarrow\infty} = 0$, $\langle\sigma\rangle_{r\rightarrow\infty} = +f$.

It is known that in the presence of the pion-nucleon interaction it is not possible to have radial or spherically symmetric solutions which are eigenstates of the total angular momentum $\vec{J}, J_z$, and isospin $\vec{\tau}, \tau_z$. We shall specialize to the so-called hedgehog[6,7] solution which stipulates a pion field of the form $\underline{\pi} = \pi(r)(\underline{\hat{e}}_r)$, i.e., links isospin to space. It is, then, not surprising that $\vec{J}, J_z$, and $\vec{\tau}, \tau_z$ do not individually commute with the Hamiltonian. However, $\vec{I} = \vec{J} + \vec{\tau}/2$ does commute and we can classify solutions as eigenstates of $\vec{I}$.

The lowest-energy normalized eigenstate is then given by the wave $l = 0$, $I = 0$ and may be written in terms of upper and lower components,

$$\psi_0 = \begin{pmatrix} if(r) \\ \vec{\sigma}\cdot\hat{r}g(r) \end{pmatrix} |v\rangle,$$

with $|v\rangle = (1/\sqrt{2})(|\uparrow\rangle|-\tfrac{1}{2}\rangle - |\downarrow\rangle|+\tfrac{1}{2}\rangle)$, where the arrows and $|\tfrac{1}{2}\rangle$ refer to spin and isospin, respectively, and $I|v\rangle = 0$; also $\int d^3r\,[f^2(r) + g^2(r)] = 1$.

From the field equations for the nucleon and meson fields it follows that [with $\varphi(r) = \sigma(r)/f_\pi$ and $\chi(r) = \pi(r)/f_\pi$]

$$g(r) \rightarrow 0 \text{ as } r \rightarrow 0; \quad \chi(r) \rightarrow 0 \text{ as } r \rightarrow 0, \tag{4}$$

For simplicity I shall impose the nonlinear constraint $\sigma^2 + \underline{\pi}^2 = f_\pi^2$ or $\lambda \rightarrow \infty$ over all space. This is, of course, true asymptotically (later, I shall show that this is reasonable).

We now make the choice $\varphi = \cos\theta(r)$ and $\chi = \sin\theta(r)$ and cast the equation in a dimensionless form in terms of $x = gf_\pi r$. Also, we switch to $G(r)/r = g(r)$ and $F(r)/r = f(r)$. The Dirac equation in matrix form is then given by[8]

$$\begin{bmatrix} \cos\theta & -(\partial_x + x^{-1}) - \sin\theta \\ \partial_x - x^{-1} - \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} = E \begin{bmatrix} F \\ G \end{bmatrix}, \tag{5}$$

where $E = e_0/gf\pi$ and

$$E(\theta) = \frac{\int^\alpha dx\,[\cos\theta(F^2 + G^2) - 2\sin\theta\, FG - F(\partial_x + x^{-1})G + G(\partial_x - x^{-1})F]}{\int (F^2 + G^2)\,dx}. \tag{6}$$

The total energy is then given, in units of gf_π, by

$$\frac{E}{gf_\pi} = E + \frac{4\pi}{g^2}\frac{1}{2}\int_0^\alpha dx \left[x^2 \left(\frac{d\theta}{dx} \right)^2 + 2\sin^2\theta \right]. \tag{7}$$

Since the σ and $\underline{\pi}$ fields are now expressed in terms of a single parameter $\theta(r)$, we shall minimize the total energy, E, with respect to $\theta(r)$, subject to the foregoing boundary conditions, instead of solving the two variational equations for σ and $\underline{\pi}$.

Note also that the meson part of the Hamiltonian or energy is exactly what follows from $H_M^0 = \tfrac{1}{2}f_\pi^2 \times \int \mathrm{Tr}(\vec{\nabla}u^\dagger\cdot\vec{\nabla}u)$, where $u = \cos\theta + i(\underline{\tau}\cdot\underline{\hat{e}}_r)\sin\theta$. H_M^0, of course, is the Hamiltonian corresponding to the Lagrangian L_0 in Ref. 4. The addition to the static meson Hamiltonian arising from the extra term, L_1,[4] needed to achieve a stable soliton in the pure meson sector is quartic in u and is given by

$$H_M^2 = \frac{1}{32e^2}\int d^3x\,\mathrm{Tr}[(\partial_i u)u^\dagger,\ (\partial_j u)u^\dagger]^2.$$

The function $\theta(x)$ must satisfy the following conditions:

$\varphi = \cos\theta = +1, \; x \to \infty;$

$\chi = \sin\theta = 0, \; x \to \infty;$

$\chi = \sin\theta = 0, \; x \to 0.$

The variation of θ which gives rise to a minimum in the energy furthermore has the form

$$\theta(x \to 0) = \pi \text{ and } \theta(x \to \infty) = 2\pi.$$

A variation of $\theta(x)$ with the above properties is performed to give the exact energy minimum. A simple and not inaccurate parametrization of $\theta(x)$ is as follows: $\theta(x)$ rises linearly from π at $x=0$ to 2π at $x=X$ and remains constant and equal to 2π thereafter. X is then a size parameter for the system. The meson energy can now be solved for analytically. The single-particle valence-nucleon bound-state energy is solved for numerically. We then have

$$\frac{E}{gf_\pi} = E_0 + (X/g^2)4\pi(\tfrac{1}{2} + \pi^2/6). \qquad (8)$$

A numerical fit to E_0 is found to have the form[8]

$$E_0 = -0.94 + 3.16/X. \qquad (9)$$

This is reasonable is its X dependence, X in some sense being the width of the "potential well" seen by the nucleon. However, there is the puzzling feature that as we change $X(g)$ to very large values the energy can go through zero and become negative. This would seem to imply an instability or inconsistency. However, the negative- and positive-energy spectra are no longer symmetric in the presence of the pion field (e.g., Coulomb field) and if we follow the negative-energy states as the valence bound-state energy goes negative, there is always a *gap* between the lowest valence state and the highest negative or sea state. There is no crossing of valence and sea states. This indicates that the system is not unstable to pair production. Also, when the total energy E and not just E_0 goes negative we have an essential instability in the theory (see Ref. 7 where exactly similar considerations are needed in a different context to bind quarks to give a chiral soliton model for the nucleon).

We now solve analytically for the one-nucleon bound state,

$$\frac{E}{gr_0} \simeq -0.94 + \frac{3.16}{X} + \frac{26.96}{g^2}X. \qquad (10)$$

Minimizing with respect to X, we find $X^2 = 3.16g^2/26.96$. This gives a value for the minimum of $E/gf_\pi \simeq 0.91$ and $X \simeq 3.6$ for the usual plane-wave nucleon mass $M_N = gf_\pi = 938$ MeV which gives $g \simeq 10$ for $f_\pi = 93$ MeV. Substituting these values, we find $E \simeq 840$ MeV and $R = X/f_\pi g \simeq 0.7$ fm. Let me add that an exact variational procedure for determining $\theta(r)$ gives $E \sim 700$ MeV.

Finally, in relaxing the nonlinear constraint of $\sigma^2 + \underline{\pi}^2 = f_\pi^2$, or more specifically, in going from $\lambda = \infty$ to $\lambda = 8$, there is hardly any change in the energy, showing the $\lambda \to \infty$ *Ansatz* to be good. The reason for this is that the system always likes to stay on the continuous ring of minima given by $\underline{\pi}^2 + \sigma^2 = f_\pi^2$ as one goes from the origin to infinity to avoid the positive volume energy that arises in departing from this.

I have shown that a bound nucleon in a hedgehog configuration of the $(\underline{\pi}, \sigma)$ fields in the σ-model chiral Lagrangian exists, with total energy less than that of a free nucleon in the usual plane-wave configuration $m_N = gf_\pi = 938$ MeV. However, the Dirac wave function is unusual, since it is not an eigenfunction of spin and isospin. At the level of expectation values we get a spin $=0$ for line $I=0$ state. I shall defer discussion on all but the baryon numbers of the object.

The naive expectation of unit baryon number, B, corresponding to a single bound nucleon is not borne out, for it has been shown[2,3] that baryon number can arise out of the nontrivial topology of the σ and $\underline{\pi}$ fields. Goldstone and Wilczek have shown that fermion (baryon) number, in a theory of nucleon-$\underline{\pi}$-σ interactions which is precisely mine, can arise from a hedgehog configuration of $\underline{\pi}$ and σ fields by themselves, that is, in the absence of any fermions in the system. But we know that in the zero-nucleon sector the present *meson* Lagrangian cannot support a soliton solution: it would shrink to the origin as there is no length scale in the problem and $E \propto R$. It is precisely the function of the additional term, L_1, in the effective Lagrangian used by Skyrme[2] and Balachandran *et al.*[4] to stabilize this solution. Let us then compare their solution with the present one.

The nonlinear σ model has a topological number associated with it corresponding to the mapping of the compactified configuration space R_3 into the $(\underline{\pi}, \sigma)$ field space (S^3) given by [Ref. 3, Eq. (6), and Ref. 4][9]

$$t = (1/12\pi^2)\int (d^3x/|\phi|^4)\epsilon^{ijk}\epsilon_{dabc}\phi^d\partial_i\phi^a\partial_j\phi^b\partial_k\phi^c$$
$$= \pi^{-1}[\theta(r=\infty) - \theta(r=0) + \tfrac{1}{2}\{\sin 2\theta(r\to\infty) - \sin 2\theta(r\to 0)\}], \qquad (11)$$

In Ref. 4 the meson Lagrangian, namely, L_0+L_1, supports a soliton with $t=1$. Goldstone and Wilczek[3] have shown that in this case "t" is also exactly the baryon number of the vacuum. By vacuum is meant the soliton background in the absence of any valence nucleon. In the soliton background the Dirac spectrum had bound states. If we occupy the lowest positive energy bound state, we get a "soliton nucleon"[10] system with $B=2$. (In Ref. 4 the lowest bound-state energy is simply added to the soliton energy to get the mass of this object, whereas the above solution is a self-consistent one which takes the reaction of the fermion source on the soliton fields into account.) Now if the term L_1 is gradually turned off, the stable solution presented above is realized. Since the meson field configuration at $x=0$ and $x=\infty$ is unchanged so is B $(=2)$. The stability of the present $B=2$ soliton (in contrast to the $B=1$ soliton) is physically understood: as the soliton shrinks, the size of the nucleon bound state pinned to it decreases, raising the nucleon kinetic energy until equilibrium is achieved. The only $B=1$ states that my solution can decay into are the plane-wave nucleons of much higher energy since the single soliton states in the present model are unstable and not physical.

The baryon number of my solution is nonperturbative. Quantum fluctuations[11] are unlikely to alter that. They will, however, change the energy as they will for the solution of Ref. 4. This question is yet to be addressed. In the absence of quantum fluctuations and the context of the well worn and accepted σ model I have found a stable object which obviously cannot decay into the two-plane-wave–nucleon channel, its energy being less than even a single nucleon. Does this uncover a problem in the accepted σ model? Certainly, nuclear physics would seem to exclude the existence of such a state. The only way to reclaim the physics is then to look back at the assumption of the model field theory which treats the nucleon as a point object in interaction with the σ and $\underline{\pi}$ fields. Clearly if the nucleon has *substructure* at the length scale of the equilibrium size ($r=0.7$ fm) of the soliton, the model field theory, as such, is no longer viable. Such considerations will apply to field theories with nontrivial topology linked to fermion number in general.

The author acknowledges many exchanges and collaboration with G. Ripka with whom a similar calculation was performed in a different context (Ref. 8). Some of the results used here are from that work. The author thanks M. Rho and J. P. Blaizot for discussions.

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[9]Here ϕ^0 is the σ field and $\phi^{1,2,3}$ the three-component pion field.

[10]In this σ model, there is thus a dual description of the baryon number or nucleon. Currently, popular *purely* mesonic Lagrangians purport to avoid this by identifying the soliton with the $B=1$ nucleon. However, any field theory of soliton (nucleon) interactions must have one pion exchange and spontaneous chiral-symmetry breaking. This would necessarily resemble the σ model, whence the soliton-nucleon duality reappears. New consistency requirements may then be necessary.

[11]There are indications that the lowest-order (to order $\hbar$) bosonic fluctuations in the linear σ model generate a term of the type L_1, the stabilizing Skyrme term [T. Applequist and C. Bernard, Phys. Rev. D **22**, 200 (1980)]. Since the fermion single-particle states give a contribution of order $\hbar$, it is just these bosonic fluctuations that we need to include to this order and they preserve the stability in the $B=1$ state.