Hausdorff Dimension and Uniformity Factor of Strange Attractors

R. Badii and A. Politi
Istituto Nazionale di Ottica, I-50125 Firenze, Italy
(Received 29 November 1983)

The Hausdorff dimension D_0 of a strange attractor is argued to be the fixed point of a recursive relation, defined in terms of a suitable average of the smallest distances δ_i between points on the attractor. A fast numerical algorithm is developed to compute D_0. The spread λ in the convergence rates towards zero of the distances δ_i (uniformity factor) as well as the stability of the fixed point are discussed in terms of the entropy of the δ_i distribution.

PACS numbers: 02.50.+s, 47.25.−c, 52.35.Ra

A peculiar feature of a strange attractor which distinguishes deterministic chaos from a random process is its fractal (Hausdorff-Besicovitch) dimension D_0.[1] A direct estimate of D_0 is usually achieved through the computation of the so-called capacity, by means of the well-known box-counting algorithm. As has been pointed out by some authors,[2,3] this is a method very consuming of memory and it is impractical for a phase-space dimension greater than two.

In order to overcome these difficulties, several different methods have been recently proposed. The first one[4] yields a quantity ν which has been shown to be a lower bound for D_0. More recently Termonia and Alexandrovitch[5] have defined a different dimension D_F, which apparently is a closer approximation to D_0.

Hausdorff dimension calculations employ collections of balls that cover the attractor, and the ball size can vary from place to place. While the definition of Hausdorff dimension requires consideration of all such collections, it is generally believed that any natural choice of ball sizes will give the same results for typical attractors. Capacity for example uses covers with all balls of the same size.

One limitation of capacity is that it is difficult to be sure how much of the low-density parts of the attractor are missed by the collection of balls. In our case it is expected that the fraction covered is constant, and this aspect becomes part of the scaling process.

As in the above mentioned methods, we start generating n points of the attractor but, differently, we study the dependence of the "mean" nearest-neighbor distance on n. By doing so, we are able to define a dimension which depends on the way the average is performed. We then show that the Hausdorff-Besicovitch dimension coincides with a specific averaging rule and, moreover, it turns out to be the fixed point of a suitable recursive relation, whose stability is proven. Furthermore we develop a practical numerical algorithm which is as fast as the methods of Refs. 4 and 5, and has nearly the same memory requirements. Finally we show that the stability coefficient λ of the fixed point D_0 directly measures the spread in the rate of convergence to zero of the nearest-neighbor distances. Indeed, this spread is the reason for the differences among the methods so far proposed. The "uniformity factor" λ can be written in terms of a suitable entropy of the distribution of mutual nearest-neighbor distances.

Let us first define ${}_1\delta_i(n)$ as the distance between the ith point and its nearest neighbor, which explicitly depends on the number of points n. The mean distance

$$ {}_1\bar{\delta}(n) = \frac{1}{n}\sum_{i=1}^{n} {}_1\delta_i(n), \qquad (1)$$

can be heuristically shown to go as

$$ {}_1\bar{\delta}(n) = Kn^{-1/D} \qquad (2)$$

in any "regular" attractor. In fact, if we fill a bounded region in d-dimensional Euclidean space with n randomly distributed points, Eq. (2) holds with $D = d$.

To make a connection with Ref. 5, we now define ${}_q\bar{\delta}(n)$ as the mean distance between qth nearest neighbors. At variance with Ref. 5, we investigate the dependence of ${}_q\bar{\delta}(n)$ on n rather than on q.

Let us introduce a generalized averaging procedure

$$ {}_q\bar{\delta}(n) \equiv \left[\frac{1}{n}\sum_{i=1}^{n} {}_q\delta_i^{\gamma}(n)\right]^{1/\gamma} = K_1(\gamma)\,n^{1/D(\gamma)} \qquad (3)$$

for some $K_1(\gamma)$ where, depending on γ, the role of different length scales is enhanced. Hence, different choices of γ yield different estimates $D(\gamma)$. In contrast, the role of q turns out to be irrelevant. Therefore, from now on, we will drop the index q, unless explicitly stated.

In order to clarify the meaning of the function $D(\gamma)$, we discuss the simple, analytically solvable generalized Baker transformation,[6] before going to the general treatment. If we indicate with α_1 and α_2 the two contraction rates, we recall that the Hausdorff dimension of the set of points lying on the x axis appears to satisfy the following relation

$$1=\alpha_1^{-D_0}+\alpha_2^{-D_0}. \quad (4)$$

Let us evaluate $D(\gamma)$. Starting with the segment (0,1), we obtain, after the first step, two intervals of lengths α_1^{-1} and α_2^{-1}. Asymptotically we are left with a Cantor set of points. In order to perform the calculation, we consider the kth step of the process and let k go to infinity. Since the extrema of the 2^k intervals are points of the asymptotic attractor, the lengths of such intervals are exactly the first nearest-neighbor distances in the set of 2^{k+1} points.[7] Hence

$$_1\bar{\delta}(n)=\left[\frac{1}{2^k}(\alpha_1^{-\gamma}+\alpha_2^{-\gamma})^k\right]^{1/\gamma}$$

$$=K(\gamma)n^{-1/D(\gamma)}, \quad (5)$$

where $n=2^{k+1}$. From this

$$D(\gamma)=\gamma\ln 2/[\ln 2-\ln(\alpha_1^{-\gamma}+\alpha_2^{-\gamma})]. \quad (6)$$

By comparing Eqs. (4) and (6) it is readily seen that, if $\gamma=D_0$, Eq. (6) becomes an identity. It is therefore natural to consider Eq. (6) as a recurrence whose fixed point is $\gamma=D_0$.

This is a very simplified example of fractal behavior. Indeed, any time a new point is generated on the attractor, the contraction rates α_1 and α_2 have the same occurrence probability. Therefore we can generalize Eq. (5), associating with α_1 and α_2, two different probabilities p_1 and $p_2=1-p_1$, respectively. The sum in Eq. (3) now cannot be analytically done; anyhow, as shown in Fig. 1 where $D(\gamma)$ is plotted for different p_1 and p_2, the fixed point D_0 remains unchanged. This is in accordance with the Hausdorff definition of dimension, since D_0 is a purely geometrical quantity which depends only on the contraction rates, but not on their respective probabilities.[8]

Incidentally the curve a in Fig. 1 corresponds to a uniform attractor, while b corresponds to the above mentioned simplified example ($p_1=p_2=\frac{1}{2}$).

We can immediately exploit the stability of the fixed point to develop a computational algorithm. Starting with a suitable value of γ, we let the system evolve and plot $\ln[\bar{\delta}(n)]$ vs $\ln(n)$, determining the new value γ' from the slope of the curve. Such a γ' is used to repeat the procedure until we get D_0 with a satisfactory accuracy. Notice that it is not necessary to use a large number of points in the first steps of the process.

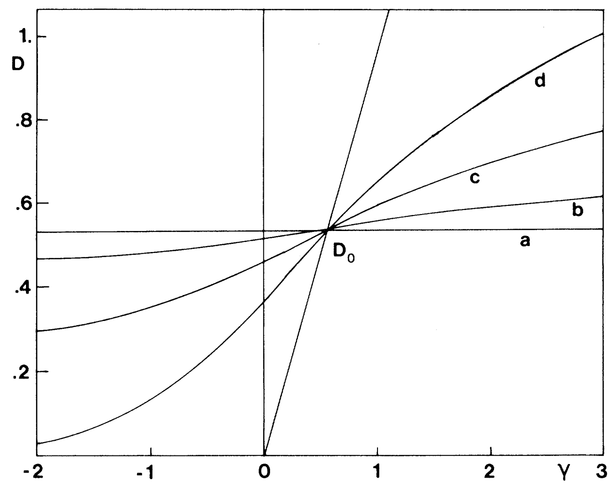

FIG. 1. Plot of the function $D(\gamma)$ vs γ in the case of the generalized Baker transformation for the values $\alpha_1=2.50\ldots$, $\alpha_2=5.80\ldots$ and different p_1,p_2. The slanted straight line is the diagonal $D=\gamma$. The curve a corresponds to $p_1=0.61$ (uniform attractor), b to $p_1=\frac{1}{2}$ (logistic map), c to $p_1=0.39$, and d to $p_1=0.25$. All the curves cross the diagonal in the same fixed point D_0.

The stability of the fixed point is simply studied by taking the derivative of Eq. (6) with respect to γ for $\gamma=D_0$,

$$\lambda\equiv D'(D_0)$$

$$=1-D_0\frac{\alpha_1^{-D_0}\ln\alpha_1+\alpha_2^{-D_0}\ln\alpha_2}{\ln 2}. \quad (7)$$

We find that, for the logistic map,[9] the uniformity factor λ is $\lambda=0.0358\ldots$, i.e., the fixed point is very stable. Indeed, if we start with $\gamma=1$ (linear average), in only two steps we get 0.5376, which differs by 0.1% from the right solution.

We now deal with the case of a general dynamical system and, for this purpose, we recall the definition of Hausdorff-Besicovitch dimension.[1] Let us consider a covering of the attractor with d-dimensional cubes of different sizes ϵ_i. We define $L_\gamma(\epsilon)$ as

$$L_\gamma(\epsilon)\equiv\inf\textstyle\sum_i\epsilon_i^\gamma, \quad (8)$$

where the infimum is taken over all coverings satisfying the condition $\epsilon_i\leq\epsilon$. When we let ϵ go to zero, there is one critical value $\gamma=D_0$ above which $L_\gamma(\epsilon)$ goes to zero and below which $L_\gamma(\epsilon)$ tends to infinity.

We rewrite Eq. (3) as

$$\bar{L}_\gamma[\delta(n)]\equiv\sum_{i=1}^{n}\delta_i^\gamma(n)=K(\gamma)n^{1-\gamma/D(\gamma)}. \quad (9)$$

We can now interpret the δ_i's as the edges of cubes centered around the n points. Such a set of cubes reasonably covers a constant fraction of the attractor (this is certainly true if the attractor is eventually self-similar). Moreover we can notice that the prescribed limit $\epsilon \to 0$ of Eq. (6) is simply accomplished by letting n tend to infinity, since all the points are recurrent. Hence our definition of dimension is reasonably equivalent to the Hausdorff-Besicovitch one. The quantity $\tilde{L}_\gamma$ has the same property as L_γ: for $\gamma < D_0$ ($> D_0$), when $D(\gamma) > \gamma$ ($< \gamma$), $\tilde{L}_\gamma$ will diverge to ∞ (converge to 0) for n tending to ∞, which is equivalent to the limit $\epsilon \to 0$ of Eq. (8). A more rigorous treatment of this point will be developed elsewhere.[10]

The general proof of the stability of the fixed point D_0 is accomplished by taking the logarithm of both sides of Eq. (3) and taking the derivative with respect to γ. We obtain

$$\frac{1}{\gamma^2}[\ln n - H_n(\gamma)] = K' + \frac{D'(\gamma)}{D^2(\gamma)}\ln n, \tag{10}$$

where

$$H_n(\gamma) = -\sum_{i=1}^{n} p_i \ln p_i, \tag{11}$$

and $p_i = \delta_i^\gamma / \sum_{l=1}^{n} \delta_l^\gamma$. This δ entropy is a positive quantity which attains its maximum value $\ln(n)$ for a uniform distribution of points over the attractor. In general it can be written as

$$H_n(\gamma) = \sigma(\gamma)\ln n, \tag{12}$$

where $0 \leqslant \sigma(\gamma) \leqslant 1$. By inspection of Eqs. (10) and (12) we find

$$\lambda \equiv D'(D_0) = 1 - \sigma(D_0). \tag{13}$$

Therefore, the fixed point $\gamma = D_0$ is stable. In particular for a uniform distribution of points it is superstable being $\sigma(D_0) = 1$. Hence, the factor λ is not only a stability parameter but also a quantitative estimate of the probabilistic character of the strange attractor.

We report in Table I the computed values of D_0 and λ for various attractors.[11] For the logistic map we obtain a value of λ in close agreement with the approximate prediction of Eq. (8) ($\lambda = 0.0358\ldots$). Our value for D_0, 0.538, is perfectly in agreement with the theoretical prediction,[13] while an application of the method in Ref. 5 yields 0.551.... We are also able to give a rough estimate λ for the Henon attractor, while for the Kaplan-Yorke map we only present an upper bound (being $\lambda \approx 0$): this explains the concordance among all the methods. The Zaslavskij map requires a detailed analysis which will be performed elsewhere. Here we simply want to notice that the Kaplan-Yorke equation,[14] which links the Lyapunov exponents with the fractal dimension, yields 1.551 ± 0.0005,[15] while our method gives 1.58 ± 0.04. Hence, differently from what is stated in Ref. 5, the value of $\nu \approx 1.50$,[4] being smaller than 1.55, is in agreement with the estimated value of D_0. Finally we point out that the Zaslavskij map with these parameter values requires a large number of points ($> 100\,000$) in order to have a first reliable estimate of D_0. This difficulty has made the measure of λ unfeasible.

Finally, our method allows the computation not only of D_0, but also of all the generalized Renyi dimensions D_q.[16] More precisely, we show in Ref. 10 that each D_q depends on different choices of γ; hence the rate of variation of D_q vs q depends on λ. In this sense knowledge of a single Renyi dimension is not sufficient to characterize a strange attractor and one must add at least the uniformity factor λ as first necessary information about the "strangeness."

The authors owe strong thanks to F. T. Arecchi for critical remarks as well as for his encouragement. One of us (A.P.) is grateful to A. Vulpiani for early discussions.

TABLE I. Computed values of D_0 and λ for various attractors.

Model[a]	D_0	Uniformity factor	Number of points
Logistic map	0.538 ± 0.001	0.038 ± 0.002	16000
Henon map	1.27 ± 0.015	0.025 ± 0.012	16000
Kaplan-Yorke map	1.43 ± 0.01	$< 5\times10^{-3}$	16000
Zaslavskij map	15.8 ± 0.04		200000

[a]The parameter values are the same as in Refs. 4, 5, and 12.

[1]B. B. Mandelbrot, *Fractals: Form, Chance and Dimension* (Freeman, San Francisco, 1977).

[2]H. Greenside, A. Wolf, J. Swift, and T. Pignataro, Phys. Rev. A **25**, 3453 (1982).

[3]P. Grassberger, Phys. Lett. **97A**, 224 (1983).

[4]P. Grassberger and I. Procaccia, Phys. Rev. Lett. **50**, 346 (1983).

[5]Y. Termonia and Z. Alexandrovitch, Phys. Rev. Lett. **51**, 1265 (1983).

[6]J. D. Farmer, E. Ott, and J. A. Yorke, Physica (Utrecht) **7D**, 153 (1983).

[7]This is true only if $(1-\alpha_1^{-1}-\alpha_2^{-1} \geqslant \max(\alpha_1^{-1}, \alpha_2^{-1})$. However, when this inequality is not verified, it can be shown that Eq. (5) still holds up to a multiplicative factor (R. Badii and A. Politi, to be published).

[8]H. G. E. Hentschel and I. Procaccia, Physica (Utrecht) **8D**, 435 (1983).

[9]Choosing $\alpha_1 = 2.50$ and $\alpha_2 = 5.80$ we get a good approximation of the attractor at the accumulation point of bifurcation cascade.

[10]Badii and Politi, Ref. 7.

[11]The behavior of the third-nearest-neighbor distances turned out to be smoother than that of first and second ones.

[12]D. A. Russell, J. D. Hanson, and E. Ott, Phys. Rev. Lett. **45**, 1175 (1980).

[13]P. Grassberger, J. Stat. Phys. **26**, 173 (1981).

[14]J. L. Kaplan and J. A. Yorke, in *Functional Differential Equations and Approximation of Fixed Points,* edited by Heinz-Otto Peitten and Heinz-Otto Walther, Lecture Notes in Mathematics Vol. 730 (Springer-Verlag, New York, 1979), p. 228.

[15]The value of 1.38... of D_0 in Ref. 12 is wrong, maybe due to a factor 2 in the measure of the largest Lyapunov exponent.

[16]A. Renyi, *Probability Theory* (North-Holland, Amsterdam, 1970).