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Quantum Measurements and Stochastic Processes

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A nonlinear stochastic process is presented that, for each realization and for large times, reproduces Lüders's projection postulate. The corresponding density operator undergoes a linear evolution reproducing von Neumann's projection postulate. The violation of the Bell inequality, for instance, is described with the two apparatus acting independently on the composed system.

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For simplicity let us first consider a two-valued observable, i.e., a projector P acting on the Hilbert space of the system. Two postulates have been proposed to relate the state after a measurement of P to the initial one. One is the von Neumann postulate[1]:

$$\rho_0 \to P\rho_0 P + (1-P)\rho_0(1-P) \tag{1}$$

which relates the initial density matrix ρ_0 to the final one. Next is Lüders's postulate[2]:

$$\psi_0 \to \begin{cases} P\psi_0/\|P\psi_0\| \text{ with probability } \langle P\rangle_{\psi_0}, \\ (1-P)\psi_0/\|(1-P)\psi_0\| \text{ with probability } \langle 1-P\rangle_{\psi_0}. \end{cases} \tag{2}$$

Lüders's postulate is intended to describe individual measurements, whereas von Neumann's postulate describes ensembles. If one averages over many measurements with the same initial state ψ_0, Lüders's postulate reduces to von Neumann's:

$$\rho_0 = \psi_0\psi_0^\dagger \to \langle P\rangle_{\psi_0}\frac{P\psi_0\psi_0^\dagger P}{\|P\psi_0\|^2} + \langle 1-P\rangle_{\psi_0}\frac{(1-P)\psi_0\psi_0^\dagger(1-P)}{\|(1-P)\psi_0\|^2} = P\rho_0 P + (1-P)\rho_0(1-P). \tag{3}$$

There are fundamental differences between these postulates: the first one is linear and deterministic (for ρ_0), whereas the second one is nonlinear and nondeterministic. This raises the following questions: (i) Can both postulates be reproduced as the asymptotic solution of a single stochastic process continuous in time? (ii) Is von Neumann's postulate compatible with a unitary evolution on a larger Hilbert space? And (iii) one can ask the same question about Lüders's postulate. By compatible we mean obtainable from physically motivated limits. In this Letter we mainly address the first question, which we answer positively by presenting a stochastic process such that (a) for each realization the state vector tends asymptotically to one of the reduced states of Lüders's postulate, (b) the ratio of realizations driving the state vector to an outcome is equal to the "quantum probability," and (c) the associated density matrix undergoes a linear evolution reproducing von Neumann's postulate. In the case of simultaneous measurements we assume that the measuring instruments act independently and we find that the proposed model reproduces the proper generalizations of (1) and (2). The second question has been answered positively by several authors who derived dynamical models of von Neumann's postulate from a unitary evolution (see, for instance, Daneri, Loinger, and Prosperi[3] and Whitten-Wolfe and Emch,[4] the latter being mathematically rigorous; see also the reviews[5]). This, however, does not answer the third question. Indeed, the density matrix can evolve according to (1), without any individual following (2). To illustrate this, let us briefly consider the following stochastic differential equation[6-8]:

$$d\psi_t = -iP\psi_t \circ dk_t, \quad dk_t = \omega dt + d\xi_t, \tag{4}$$

where ξ_t is the Wiener process $[(d\xi_t)^2 = dt]$ and $\circ$ denotes the Stratonovich product.[6,8] Since P has two eigenvalues, Eq. (4) describes a fictitious spin $\frac{1}{2}$. ω is then the mean Larmor frequency and ξ_t describes a fluctuating magnetic field parallel to the static field. Let $p_t \equiv \langle \psi_t | P | \psi_t \rangle$ and $\rho_t \equiv \langle\langle \psi_t \psi_t^\dagger \rangle\rangle$ where the double brackets denote the average over the Wiener process ξ_t. From (4) one obtains $dp_t = 0$ and

$$\dot{\rho}_t = -i\omega[P, \rho_t] - \tfrac{1}{2}(P\rho_t + \rho_t P) + P\rho_t P. \tag{5}$$

Consequently, for each realization, p_t is a constant. Equation (4) thus does not reproduce Lüders's postulate. However, Eq. (5) reproduces von Neumann's postulate. Accordingly the third question remains open. But, as we shall see, the present model makes it more precise.

The model is defined by Eqs. (7), (8), and (10) below. In order to present it, let us first consider the following generalization of the Schrödinger equation:

$$\dot{\psi}_t = -iH\psi_t + k(\langle H \rangle_{\psi_t} - H)\psi_t, \tag{6}$$

where $H = H^\dagger$, $k > 0$, and

$$\langle H \rangle_{\psi_t} = \langle \psi_t | H | \psi_t \rangle / \langle \psi_t | \psi_t \rangle.$$

This nonlinear evolution equation has been studied by Gisin.[9,10] It is the most general with the following property[10]: If $\dot{\phi}_t = -iH\phi_t - kH\phi_t$ and $\psi_t \equiv \phi_t / \| \phi_t \|$, then ψ_t follows Eq. (6) and all solutions of (6) are of this form. Accordingly, the nonlinearity of Eq. (6) is of the same kind as the one which occurs in Lüders's postulate (2). Let us note that if H is bounded below, then, by Sz.-Nagey's theorem,[11] the evolution (6) can be dilated to a unitary evolution on a larger Hilbert space.[12] From (6) one obtains

$$(d/dt)\langle H \rangle_{\psi_t} = -2k(\langle H^2 \rangle_{\psi_t} - \langle H \rangle_{\psi_t}^2) \leq 0.$$

The system thus dissipates energy, except when it is in an eigenstate of H, and it tends asymptotically toward a stationary state. However, only the ground state is stable.[9]

Accordingly Eq. (6) describes the action of a heat bath at zero temperature. The situation we want to describe differs from this in two respects. First, as it takes place at finite temperature, it is reasonable to assume that the coefficient k is a stochastic process. The Hamiltonian term could also contain fluctuating terms, but they would not affect the relaxation mechanism we are interested in. Next, a measurement apparatus differs from a heat bath by the fact that its state evolves in relation to the state of the system. I assume thus that ψ_t and k_t satisfy coupled stochastic differential equations[6-8]:

$$d\psi_t = -i\omega P\psi_t dt + (p_t - P)\psi_t \circ dk_t, \tag{7}$$

$$dk_t = f(k_t, \psi_t)dt + d\alpha_t, \tag{8}$$

where $\omega \in \mathbf{R}$; $p_t \equiv \langle P \rangle_{\psi_t}$; α_t is the Wiener process $[(d\alpha_t)^2 = dt]$; $\circ$ denotes the Stratonovich product[6,8]; and f is a function which we shall choose such that the average of p_t over α_t is constant. As we shall see, this is necessary for the description of measurements which overlap in time. Upon recalling the relation between Stratonovich and Itô products,[6,8] $X \circ dY = XdY + \frac{1}{2}dXdY$, we obtain from (7) and (8)

$$dp_t = 2p_t(1 - p_t)\{[1 - 2p_t - f(k_t, \psi_t)]dt + d\alpha_t\}. \tag{9}$$

Hence, the only possible choice for f is

$$f(k_t, \psi_t) = 1 - 2p_t. \tag{10}$$

Equations (7), (8), and (10) define the model. Each realization describes a single measurement. The values that the stochastic process assumes then are interpreted as part of the state of the apparatus. Note that since f depends on p_t, Eq. (7) is not of the type studied in Ref. 10, but is a generalization of Gisin and Piron.[13]

From (9) and (10) one sees that the average of p_t is time independent:

$$\langle\langle p_t \rangle\rangle = \langle\langle p_0 \rangle\rangle = \langle P \rangle_{\psi_0}, \tag{11}$$

but for each realization of α_t, p_t is time dependent. Let z_t be defined by $\tanh z_t \equiv 2p_t - 1$. From (9) and (10) one obtains

$$dz_t = \tanh z_t \, dt - d\alpha_t. \tag{12}$$

The associated Fokker-Planck equation for the distribution function $\lambda_t(z)$ reads

$$\dot{\lambda}_t(z) = -\partial_z[\tanh(z)\lambda_t] + \tfrac{1}{2}\partial_z^2\lambda_t. \tag{13}$$

The solution of (13) is known[14] and provides the distribution function $\rho_t(p)$ corresponding to (9):

$$\lambda_t(z) = (2\pi t)^{-1/2}\exp\left\{-\frac{t}{2} - \frac{(z-z_0)^2}{2t}\right\}\frac{\cosh z}{\cosh z_0}, \quad \rho_t(p) = \lambda_t[\tanh^{-1}(2p-1)]/2p(1-p). \tag{14}$$

ρ_t is normalized, but $\rho_t(p) \to 0$ for $t \to \infty$, $p \neq 0, 1$. Consequently, the distribution ρ_t concentrates asymptotically at the points $p=0$ and $p=1$, and from (11) one sees that the weights of these concentrations are precisely the "quantum probabilities" $\langle P\rangle_{\psi_0}$ and $\langle 1-P\rangle_{\psi_0}$. Finally the study of equations of the form (7) [or (6)] shows that for each realization of the stochastic process k_t, the state vector ψ_t moves in the plane defined by ψ_0 and $P\psi_0$[9] [if $P\psi_0=0$ or $P\psi_0=\psi_0$, then ψ_0 is a stationary solution of (7)].

Consequently, each realization of k_t reduces the state vector ψ_0 as postulated by Lüders and the ratio of realizations of k_t which drive ψ_0 toward $P\psi_0/\|P\psi_0\|$ is equal to $\langle P\rangle_{\psi_0}$.

Let us now consider the density operator which represents the average over k_t of the one-dimensional projector $\psi_t\psi_t^\dagger$: $\rho_t = \langle\langle \psi_t\psi_t^\dagger\rangle\rangle$. Equation (7) is highly nonlinear. A straightforward computation shows, however, that one recovers the simple linear equations (5). The stochastic process (7) mimics thus both projection postulates.

I now generalize the model to simultaneous measurements of n compatible projectors $P_1, \ldots, P_n$, $[P_i,P_j]=0\ \forall\ i,j$. For example, each P_j could represent a counter, or a sensitive center of a screen. For a one-particle system the P_j's would then be mutually orthogonal (i.e., $P_iP_j=\delta_{ij}P_j$), but this is not necessary for two-particle systems, as, for instance, the famous Einstein-Podolsky-Rosen correlated particles.[15,16]

A natural extension of the model presented above consists in adding n similar terms to Eq. (7) with n independent noises. The idea is that the n apparatus act independently and each apparatus acts in the same way whether or not there are other apparatus.

$$d\psi_t = \sum_{j=1}^{n}\{-i\omega_jP_j\psi_t\,dt + [p_j(t)-P_j]\psi_t\circ dk_j(t)\}, \quad dk_j(t) = [1-2p_j(t)]dt + d\alpha_j(t), \tag{15}$$

where $p_j(t) = \langle P_j\rangle_{\psi_t}$ and the $\alpha_j(t)$'s are n independent Wiener processes. I do not know the solution of (15). I shall nevertheless prove that Eq. (15) reproduces Lüders's postulate for each realization of the n stochastic processes α_j, and von Neumann's postulate for the density operator $\rho_t = \langle\langle\psi_t\psi_t^\dagger\rangle\rangle$.

Put $p_{i_1\ldots i_e} = \langle P_{i_1}\cdots P_{i_e}\rangle_{\psi_t}$. From (15) one obtains

$$dp_{i_1\ldots i_e} = 2\sum_{j=1}^{n}(p_jp_{i_1\ldots i_e} - p_{ji_1\ldots i_e})d\alpha_j, \tag{16}$$

$$\frac{d}{dt}\langle\langle p_e(1-p_e)\rangle\rangle = -4\sum_{j=1}^{n}\langle\langle (p_jp_e-p_{je})^2\rangle\rangle.$$

The average of $p_e(1-p_e)$ is thus a nonincreasing function of time bounded below by 0. Consequently its time derivative tends to 0. Hence the distribution function concentrates for large times on the state vectors for which $p_jp_e=p_{je}\ \forall\ j,e$. If $j=e$ this implies that for each realization of the stochastic processes α_j one has

$$p_j(t) \underset{t\to\infty}{\longrightarrow} 0 \text{ or } 1 \text{ for all } j=1,\ldots,n. \tag{17}$$

Now, if the P_j's are mutually orthogonal, at most one $p_j(t)$ may tend to 1. But in general, several $p_j(t)$'s may tend to 1. Let $\{Q_i\}$, $i=1,\ldots,m$, be the complete set of mutually orthogonal projectors such that $\{Q_i\psi_0/\|Q_i\psi_0\|\}$, $i=1,\ldots,m$, is the set of all possible outcomes (i.e., $\{Q_i\}$, $i=1,\ldots,m$, is the spectral family of the operator $\sum_{j=1}^{n}2^{-j}P_j$). The Q_i's are products of P_j's and $(1-P_j)$'s. Hence, from (16) and (17) gets

$$\frac{d}{dt}\langle\langle\psi_t^\dagger Q_i\psi_t\rangle\rangle \underset{t\to\infty}{\longrightarrow} 0 \text{ or } 1\ \forall\ i=1,\ldots,m.$$

Finally, Eq. (15) is such that the state vector ψ_t evolves in the vector space generated by $Q_1\psi_0,\ldots,Q_m\psi_0$. Consequently

$$\text{Prob}\left[\psi_0 \underset{t\to\infty}{\longrightarrow} \frac{Q_i\psi_0}{\|Q_i\psi_0\|}\right] = \langle Q_i\rangle_{\psi_0}$$

for all $i=1, \ldots, m$, and Lüders's postulate is satisfied.

Let us now consider the density operator $\rho_t = \langle\langle \psi_t \psi_t^\dagger \rangle\rangle$. A straightforward computation shows that one obtains a linear evolution equation ($\{\ldots,\ldots\}$ denotes the anticommutator):

$$\dot{\rho}_t = \sum_{j=1}^{n} (-i\omega_j [P_j, \rho_t] - \tfrac{1}{2}\{P_j, \rho_t\} + P_j \rho_t P_j). \tag{18}$$

The solution of (18) is easily computed with the help of the "matrix elements" $Q_i \rho_t Q_j$. It shows that for large times ρ_t satisfies von Neumann's postulate.

To conclude let me make four comments:

(1) Although the stochastic process (15) is nonlinear, the evolution of the density operator is linear, in accordance with Lüders's and von Neumann's postulates.

(2) The apparatus act independently of each other and the projectors which represent them need neither to be mutually orthogonal, nor to form a complete set. The model can thus describe the experimental tests of the famous Einstein-Podolsky-Rosen-Bohm correlations.[15,16] The fact that the model reproduces quantum mechanics, and thus violates Bell's inequality,[17,18] is possible because of the nonlocality of the evolution (7). Note that if two measurements do not exactly coincide in time, the model's predictions still agree with quantum mechanics.

(3) If one interprets the state vector as a representation of the complete state of an individual quantum system, then one should study evolutions which reproduce Lüders's projection postulate, as has been emphasized, among others, by Pearle.[19] Let us note here "individual systems" are opposed to "elements of statistical ensembles"; this does thus not exclude composed systems.

(4) The Wiener process of α_j of the model plays the same role as in the well-known model of Brownian motion of a classical particle. This suggests that there is a—yet unknown—underlying deterministic evolution, as in the classical Ford-Kac-Mazur model.[20] The underlying evolution, however, would be very sensitive to the precise microscopic state of the system plus apparatus. Note that this is neither a hidden-variable theory (compare with Bohm and Bub[21]), nor does it contradict the fact that experiments, defined in terms of macroscopical apparatus, do mostly not have predetermined outcomes.

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[1]J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton Univ. Press, Princeton, N.J., 1955).

[2]G. Lüders, Ann. Phys. (Leipzig) **8**, 322 (1951).

[3]A. Daneri, A. Loinger, and G. M. Prosperi, Nucl. Phys. **33**, 297 (1962).

[4]B. Whitten-Wolfe and G. G. Emch, Helv. Phys. Acta **49**, 45 (1976).

[5]G. Reece, Int. J. Theor. Phys. **7**, 81 (1973); K. Hepp, Helv. Phys. Acta **45**, 237 (1972).

[6]K. Itô, Memoirs Am. Math. Soc., No. 4 (1951). See also *International Symposium on Mathematical Problems in Theoretical Physics,* edited by H. Araki, Lecture Notes in Physics Vol. 39 (Springer-Verlag, Berlin, 1975), p. 218.

[7]L. Arnold, *Stochastic Differential Equations: Theory and Applications* (Wiley, New York, 1974).

[8]H. Hasegawa and H. Ezawa, Prog. Theor. Phys., Suppl. **69**, 41 (1980).

[9]N. Gisin, J. Phys. A **14**, 2259 (1981).

[10]N. Gisin, J. Math. Phys. (N.Y.) **24**, 1779 (1983).

[11]B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space* (North-Holland, Amsterdam, 1970).

[12]N. Gisin, Physica (Utrecht) **111A**, 364 (1982).

[13]N. Gisin and C. Piron, Lett. Math. Phys. **5**, 379 (1981).

[14]M.-O. Hongler, Phys. Lett. **75A**, 3 (1979).

[15]A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[16]D. Bohm, *Quantum Theory* (Prentice-Hall, Englewood Cliffs, 1951).

[17]J. S. Bell, Physics (N.Y.) **1**, 195 (1964).

[18]J. F. Clauser and A. Shimony, Rep. Prog. Phys. **41**, 1881 (1978); A. Aspect *et al.*, Phys. Rev. Lett. **49**, 1804 (1982).

[19]Ph. Pearle, Phys. Rev. D **13**, 857 (1976), and Int. J. Theor. Phys. **18**, 489 (1979), and Found. Phys. **12**, 249 (1982).

[20]G. W. Ford, M. Kac, and P. Mazur, J. Math. Phys. (N.Y.) **6**, 504 (1965).

[21]D. Bohm and J. Bub, Rev. Mod. Phys. **38**, 453 (1966).