Chiral Anomaly and the Rational Quantization of the Hall Conductance

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The quantum Hall effect is shown be equivalent to a chiral anomaly in quantum electrodynamics. The integers n_1 and n_2 in the rationally quantized Hall conductance $g=(e^2/2\pi\hbar)(n_1/n_2)$ arise from vortex excitations carrying n_1 electrons and n_2 flux quanta. It is found that $n_2=2j$, where $j=l+\frac{1}{2}$ is the angular momentum per electron in the vortex. This explains the odd-denominator rule $n_2=2l+1$.

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The rationally quantized Hall effect[1-3] refers to regions, in gate voltage and magnetic field, wherein the Hall conductance of the gate of a field-effect transistor has the form[4-10]

$$g=(e^2/2\pi\hbar)H, \tag{1}$$

where H is a rational number

$$H=n_1/n_2. \tag{2}$$

The purpose of this work is to discuss the physical significance, and (thereby) the possible values, of the integers n_1 and n_2.

The following model was previously shown[6] to yield Eqs. (1) and (2). If one supposes an excitation on the Hall surface which carries a charge e^* and a magnetic flux ϕ^*, via a current vortex,

$$|\text{vortex}\rangle = |e^*,\phi^*\rangle, \tag{3}$$

then the flow of such vortices (see Fig. 1) across the Hall surface yields an easily calculable Hall conductance. The source-to-drain current due to such vortex flows is evidently

$$I=e^*\,dN/dt, \tag{4a}$$

while the voltage due to flux quanta passing across a curve connecting two Hall probes is determined by Faraday's law,

$$V=(\phi^*/c)\,dN/dt. \tag{4b}$$

In Eqs. (4), dN/dt is the number of vortices flowing across the Hall surface per unit time.

The Hall conductance implied by Eqs. (4),

$$I=gV, \tag{5}$$

is evidently

$$g=c(e^*/\phi^*). \tag{6}$$

It is reasonable to assume that charge excitations arise in integral units of the electronic charge, while flux excitations arise in quantum flux units. If this is true, then (in Gaussian units)

$$e^*=n_1 e, \quad \phi^*=n_2(2\pi\hbar c/e) \tag{7}$$

imply Eqs. (1) and (2) in virtue of Eq. (6). A proof of Eq. (7), for the quantized electrodynamic field, must obviously proceed via relativistic quantum field theory. This is evident, if for no other reason, because high-precision measurements of

$$\alpha=e^2/\hbar c \tag{8}$$

require a quantum electrodynamic theoretical framework in the interpretation.

The central feature of the work which follows is

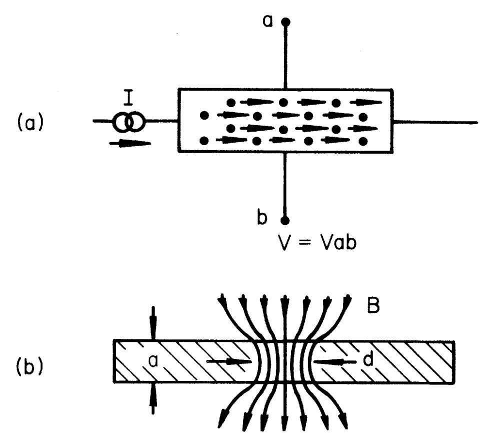

FIG. 1. (a) Vortices flowing across a Hall surface at a rate dN/dt. The current is evidently $I=e^*\,dN/dt$, and the Faraday-law voltage is $V=(\phi^*/c)dN/dt$. (b) Magnetic field lines due to a vortex on a Hall surface of thickness a. The vortex core diameter is given by $d=ac/g$.

the notion that the product of the charge and magnetic flux of a vortex excitation represents the angular momentum of the vortex (in a Lorentz frame in which the excitation vortex is at the origin),

$$e^*\phi^*/4\pi c = J^*, \tag{9}$$

so that the integers n_1 and n_2 obey the product rule

$$n_1 n_2 = 2J^*/\hbar. \tag{10}$$

Equation (10) will be considered further, after some quantum electrodynamic considerations entering into the proof of its validity are outlined.

First let us express the definition $I = gV$ as a relativistic vector relation between a conserved current vector[11]

$$J^\mu = (J_x, J_y, c\rho), \quad \partial_\mu J^\mu = 0, \tag{11a}$$

and the electromagnetic conserved pseudovector

$$f^\mu = (E_y, -E_x, B), \quad \partial_\mu f^\mu = 0, \tag{11b}$$

in the quantum electrodynamic notation of two spatial dimensions plus one time dimension. The charge flow is proportional to the magnetic flux flow, since both are carried by the same vortex flow, i.e.,

$$J^\mu = g f^\mu. \tag{12}$$

Equation (12) is the local relativistic version of Eq. (5).

Let a be the "effective thickness" of the Hall surface and define the "charge" in the $(2+1)$-dimensional quantum electrodynamics so that it has conventional Gaussian units. The Hall-effect action ΔW is derived from the current

$$J^\mu = c^2 \delta \Delta W / \delta A_\mu, \tag{13}$$

where A_μ is the vector potential, and

$$f_\mu = \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda = \tfrac{1}{2} \epsilon_{\mu\nu\lambda} F^{\nu\lambda}. \tag{14}$$

ΔW can be computed from Eqs. (12)–(14) as

$$\Delta W = (g/2c^2) \int d^3x\, f^\mu A_\mu. \tag{15}$$

The electromagnetic action

$$W_0 = (a/8\pi c) \int d^3x\, f^\mu f_\mu = (a/8\pi c) \int d^3x\, (E^2 - B^2) \tag{16}$$

receives an additional Hall-effect contribution ΔW after path integration over electronic degrees of freedom, so that the total effective model action of this work is

$$W_{\text{tot}} = W_0 + \Delta W. \tag{17}$$

The angular momentum J can be computed from the action in Eq. (17) by first computing the Belinfante symmetric stress tensor via the metric.[12] For the problem at hand,

$$J = (-a/4\pi c) \int (\vec{r} \cdot \vec{E}) B\, d^2 r. \tag{18}$$

[The method used to derive Eq. (18) is more generally valid than the construction using Noether currents which is only sometimes correct.]

To compute the angular momentum J for a single vortex at the origin, one can place an external charge source e^* at the origin, and note that the radial electric field induces a circulating current, and thus a magnetic field. This can be shown to be given by

$$B^* = -(8\pi e^* g / ca^2) K_0(4\pi g r / ca), \tag{19a}$$

where $K_0(z)$ is the zeroth-order modified Bessel function.[13] The "screened" electric field can be shown to be

$$\vec{E}^* = (4\pi g/ca)^{-1} \operatorname{grad} B^*. \tag{19b}$$

The reason for the screened fields given in Eqs. (19) is that the photon propagator, implicit in the action of Eq. (17), has grown a "mass," which (in inverse length units) is given by

$$K = 4\pi g / ca. \tag{20}$$

From Eqs. (18) and (19) one easily derives Eqs. (6) and (9). This is what we wished to prove from field theory.

The angular momentum considerations are now crucial. For n_1 electrons in the vortex, it follows from Eq. (10) that the number of magnetic flux quanta is given by

$$n_2 = 2j, \tag{21a}$$

where j is the angular momentum per electron,

$$j = J^*/\hbar n_1. \tag{21b}$$

The Fermi statistics of an electron dictates that

$$j = l + \tfrac{1}{2}. \tag{22}$$

where l, be it orbital electron or photon angular momentum, must be an integer.

The central new result of this work is then

$$n_2 = 2l + 1, \tag{23}$$

so that the rational quantizations of the Hall conductance in Eqs. (1) and (2) have "odd" denominators.

It is physically evident that angular momentum arguments require rotational symmetry in the action. Thus in laboratory experiments, only "clean" surfaces should show the rational quantization effect. The "quenching" of l when impurities are present might yield only $l=0$, or integer quantizations as observable.

To conclude, we would like to explain the "chiral anomaly" and its relationship to the quantum Hall effect, particularly Eq. (15). An easy route to take starts by noting that in $3+1$ dimensions we are free to add to the classical action the term

$$\Delta W \theta \int dt \int d^3r\, (\vec{E}\cdot\vec{B}), \tag{24}$$

where θ is an arbitrary constant. We are free to add this term in classical theory, since it does not affect the equations of motion (Maxwell's equations).

We may now convert the spatial integral in Eq. (24) to a surface integral over the boundaries of the condensed matter to obtain

$$\Delta W = \tfrac{1}{2}\theta \int dt \left[\int d\vec{\Sigma}\cdot(\vec{A}\times\vec{E} - \phi\vec{B}) + dG(t)/dt \right], \tag{25}$$

where $G(t)$ is proportional to $\int d^3r\, \vec{A}\cdot\vec{B}$. The boundaries are those of a capacitor, one of which is grounded and on which the electric field and scalar potential vanish, while the other is the gate of the field-effect transistor.

Apart from the total time derivative of G (which can be regauged into the quantum electrodynamic wave function of the field coordinates), Eq. (25) can be rewritten in the form

$$\Delta W = \tfrac{1}{2}\theta \int d^3x\, f_\mu A^\mu, \tag{26}$$

where f_μ is given by Eq. (14). To do this, we make use of the relations $F_{i0}=E_i$ and $F_{12}=B_3$. We finally make contact with Eq. (15) by choosing $\theta = g/c$, where g is the same quantity as that appearing in Eq. (6). Thus the classically allowed addition to the action in $3+1$ dimensions, Eq. (24), leads to the anomaly, Eq. (26), in $2+1$ dimensions.[14]

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