Statistics of Quasiparticles and the Hierarchy of Fractional Quantized Hall States

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Quasiparticles at the fractional quantized Hall states obey quantization rules appropriate to particles of fractional statistics. Stable states at various rational filling factors may be constructed iteratively by adding quasiparticles or holes to lower-order states, and the corresponding energies have been estimated.

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Observations of the fractional quantized Hall effect[1] show that there exist special stable states of a two-dimensional electron gas, in strong perpendicular magnetic field B, occurring at a set of rational values of ν, the filling factor of the Landau level. Laughlin[2] has constructed an explicit trial wave function (product wave function) to explain the states at $\nu = 1/m$, with m an odd integer, and has argued that the elementary excitations from the stable states are quasiparticles with fractional electric charge. Among the proposals to explain the other observed fractional Hall steps are hierarchical schemes, in which higher-order stable states ν_{s+1} are built up by adding quasiparticles to a stable state ν_s of smaller numerator and denominator.[3–5]

In the present note, we observe that the quantization rules which determine the allowed quasiparticle spacings are just those that would be expected for a set of identical charged particles that obey *fractional statistics*—i.e., such that the wave function changes by a complex phase factor when two particles are interchanged. Moreover, by assuming that the dominant interaction between quasiparticles is just the Coulomb interaction between the quasiparticle charges, we are led to a natural set of approximations for the ground-state energies and energy gaps at all levels of the hierarchy.

The appearance of fractional statistics in the present context is strongly reminiscent of the fractional statistics introduced by Wilczek to describe charged particles tied to "magnetic flux tubes" in two dimensions.[6] As in Ref. 6, the quasiparticles can *also* be described by wave functions obeying Bose or Fermi statistics, the various representations being related by a "singular gauge transformation." The boson description was, in fact, used in Refs. 3 and 4 and the fermion description in Ref. 5. However, the boson or fermion descriptions require, in effect, a long-range interaction between quasiparticles which alters the usual quantization rules. The transformation between representations is analogous to the well-known transformation between impenetrable bosons and fermions in one dimension.

As in previous discussions of the fractional quantized Hall effect, we consider a two-dimensional system of electrons in the lowest Landau level, with a uniform positive background. The filling factor ν is defined by $\nu \equiv n/2\pi l_0^2$, where n is the density of electrons, and $l_0 \equiv |Be/\hbar c|^{1/2}$ is the magnetic length; hence ν is the number of electrons per quantum of flux.

Let ν_s be a stable rational filling factor obtained at level s of the hierarchy. I assert that the low-lying energy states for filling factors near to ν_s can be described by the addition of a small density of quasiparticle excitations to the ground state at ν_s. The elementary quasiparticle excitations are of two types—particlelike "p excitations" and holelike "h excitations"—having charges $q_s e$ and $-q_s e$, respectively, according to a sign convention described below. For the present purposes we need only consider states with one type of excitation present. We shall describe these states by a pseudo wave function Ψ, which is a function of the coordinates $\vec{\mathbf{R}}_k$ of the N_s quasiparticles present. I assert that the the allowed pseudo wave functions can be written in the form

$$\Psi[\vec{\mathbf{R}}_k] = P[Z_k] Q_s[Z_k] \prod_{k=1}^{N_s} \exp(-|q_s||Z_k|^2/4l_0^2), \tag{1}$$

where $Z_k \equiv X_k \mp iY_k$ is the position in complex notation, with the sign depending on the sign of the charge of the quasiparticle, $P[Z_k]$ is a *symmetric polynomial* in the variables Z_k, and

$$Q_s \equiv \prod_{k<l} |Z_k - Z_l|^{-\alpha/m_s}. \tag{2}$$

In Eq. (2), $\alpha = \pm 1$, according to whether we are dealing with particle- or hole-type excitations, and m_s is a rational ≥ 1, to be specified by an iterative equation below. We may interpret $|\Psi[\vec{\mathbf{R}}_k]|^2$ as the probability density for finding a quasiparticle at each of the positions $\vec{\mathbf{R}}_1, \ldots, \vec{\mathbf{R}}_{N_s}$, at least in the case

that the $\vec{R}_k$ are not too close to each other. Since the quasiparticles have a finite size (of order l_0), however, there is no direct significance to the behavior of $|\Psi|^2$ when two positions R_k and R_l come very close together. The wave function is normalized if $\int|\Psi|^2=1$, and two wave functions Ψ and Ψ' are orthogonal if $\int\Psi^*\Psi'=0$.

The pseudo wave function (1)–(2) can be derived in different ways, starting from various microscopic descriptions that have been proposed[2–5] for the electronic state with quasiparticle or quasihole excitations. I shall give below a derivation for p excitations using the *pair model* proposed in Ref. 3.

Because there is no direct physical significance to the phase of the pseudo wave function, it is permissible to *redefine* the factor Q in Eq. (1) by *removing the absolute value sign* in Eq. (2). (This operation may be described as a singular gauge transformation.)[6] If $m_s\neq 1$, the new wave function is a multivalued function of the positions $\{\vec{R}_k\}$, and one should consider it as a function defined on the appropriate Riemann surface for $\{Z_k\}$. [Alternatively one could use a single-valued definition and specify discontinuities along cuts in the variable (Z_k-Z_l).] Now if we continuously interchange the positions of two quasiparticles, the wave function will change by a complex phase factor $(-1)^{\pm 1/m_s}$, with the sign depending on the sense of rotation as the quasiparticles pass by each other. Although the extra phase factor is perhaps a complication, the pseudo wave function now has the esthetically pleasing property that it is an eigenstate of the differential operator $[\nabla_k\mp iq_se\vec{A}(\vec{R}_k)/\hbar c]^2$ with special boundary conditions at the points $Z_k=Z_l$, where $\vec{A}$ is the vector potential in the symmetric gauge. Then Eq. (1) may be described as a general wave function appropriate to a collection of particles of charge $\pm q_se$ obeying fractional statistics, all in the lowest Landau level. Of course, in the special case $m_s=1$, the quasiparticles are ordinary fermions.

In order to find the ground-state configuration for a given density n_s of quasiparticles, we must find the symmetric polynomial $P[Z]$ which leads to the minimum expectation value of the repulsive interaction between the quasiparticles. Using the same reasoning as Laughlin in Ref. 2, we expect that certain choices of P can lead to specially low energies, namely,

$$P[Z_k]=\prod_{k<l}(Z_k-Z_l)^{2p_{s+1}}, \tag{3}$$

where p_{s+1} is a positive integer. The probability distribution $|\Psi|^2$ is then that of a classical one-component plasma[2] with dimensionless inverse temperature $\Gamma=2m_{2+1}$, where

$$m_{s+1}=2p_{s+1}-\alpha_{s+1}/m_s, \tag{4}$$

and $\alpha_{s+1}=1$ or -1 as particlelike or holelike quasiparticles are involved. The density of the plasma is fixed by a charge neutrality condition,[2] so that the number of quasiparticles in an area $2\pi l_0^2$ is just $n_s=|q_s|/m_{s+1}$. Since each quasiparticle has charge $\alpha_{s+1}q_s$, we may readily calculate the electron density in the new stable state, and we find the filling factor

$$\nu_{s+1}=\nu_s+\alpha_{s+1}q_s|q_s|/m_{s+1}. \tag{5}$$

If we multiply the pseudo wave function described above by the factor $\prod_k Z_k$, for $k=1,\ldots,N_s$, we find a deficiency near the origin of $1/m_{s+1}$ quasiparticles of level s. We identify this state as a hole excitation at level $s+1$. Similarly, we may construct a p excitation having an *excess* of $1/m_{s+1}$ quasiparticles at the origin. The iterative equation for q_s is thus

$$q_{s+1}=\alpha_{s+1}q_s/m_{s+1}. \tag{6}$$

Together with the starting conditions $\nu_0=0$, $q_0=m_0=\alpha_1=1$, the iterative equations (4)–(6) give a sequence of rational filling factors ν_s for any choice of the sequence $\{\alpha_s,p_s\}$. At the level $s=1$, we recover Laughlin's states with $\nu_1=1/m_1=1,\frac{1}{3},\frac{1}{5},\ldots$ for various choices of p_1. If we add holes to the state $\nu_1=1$, we find at level $s=2$, the complements to the Laughlin states, $\nu_2=\frac{2}{3},\frac{4}{5},\frac{6}{7}\ldots$. (In order to stay in the lowest Landau level, we impose the restriction $\alpha_2=-1$, if $\nu_1=1$.) From the state $\nu_1=\frac{1}{3}$, we achieve such states as $\nu_2=\frac{2}{5}$ or $\frac{4}{11}$, with p excitations, and $\nu_2=\frac{2}{7}$ or $\frac{4}{13}$, with h excitations.

It can be shown, after some algebra, that the allowed values of ν_s may be expressed as continued fractions in terms of the finite sequences $\{\alpha_s,p_s\}$ and that they are identical to those of Haldane.[4] (I have used the opposite sign for α, however, and here p is one-half of Haldane's.) As noted by Haldane, every rational value of ν with odd denominator, with $0<\nu\leq 1$, is obtained once in this way. There will *not* be a quantized Hall step at *every* such rational ν, however. We know that there exists a maximum allowed value m_c for the parameter m_s, such that if at any stage of the hierarchy the calculated m_s is greater than m_c, then the quasiparticles at the density n_s will form a Wigner crystal rather than a quantum-liquid state.[2] There is then no stabilization of the electron density at the correspond-

ing ν_s, and there will be no meaning to any further states in the hierarchy constructed from this ν_s.

The pseudo wave function (1)–(3) leads to a natural estimate of the potential energy of the system, if we assume that the dependence on the positions of the quasiparticles can be approximated by the pairwise Coulomb interaction between point particles of charge $q_s e$, in the background dielectric constant ϵ. If $E(\nu)$ is the energy per quantum of magnetic flux, we have

$$E(\nu_{s+1}) \cong E(\nu_s) + n_s \epsilon_s^{\pm} + n_s |q_s|^{5/2} u_{\rm pl}(m_s), \tag{7}$$

where $\epsilon_s^{\pm}$ is the energy to add one particlelike excitation or one holelike excitation, together with neutralizing uniform background, to the state ν_s, and $u_{\rm pl}$ is a smooth function of m_s, given (approximately) by Laughlin's interpolation formula[5]

$$u_{\rm pl}(m) = \frac{-0.814}{m^{1/2}}\left(1 - \frac{0.230}{m^{0.64}}\right)\left(\frac{e^2}{\epsilon l_0}\right). \tag{8}$$

We recall that $u_{\rm pl}(m)$ is the potential energy per particle that one would find for a system of electrons at filling factor $\nu = 1/m$ if one approximates the pair correlation function $g(r)$ for the electrons by the pair correlation function $g_{\rm pl}(r)$ for a one-component plasma at inverse temperature $\Gamma = 2m$; the factor $|q_s|^{5/2}$ in the last term of (7) reflects the smaller charge and larger magnetic length for our quasiparticles.

In order to use Eq. (7), we need an iterative formula for the quasiparticle energies $\epsilon_s^{\pm}$. It is convenient to write

$$\epsilon_s^{\pm} = \tilde{\epsilon}_s^{\pm} \pm m_s^{-1}[\epsilon_{s-1} + \tfrac{3}{2}|q_{s-1}|^{5/2} u_{\rm pl}(m_s)]. \tag{9}$$

The quantity in square brackets is the energy it would take to add one quasiparticle or quasihole of level $s-1$, if one could keep the Laughlin product form (3) for the polynomial P, and simply increase the density n_{s-1} by means of a reduction, of order $1/N$, in the magnetic length l_0 which controls the distance scale in Eq. (1).[7] The term $\tilde{\epsilon}_s^{\pm}$ in (9) may be called the *proper* excitation energy; it is relatively small, but is presumably positive for both quasiparticles and holes. For the proper hole energy, we use the approximate formula

$$\tilde{\epsilon}_s^{-} = 0.313 |q_{s-1}|^{5/2} m_s^{-9/4} (e^2/\epsilon l_0). \tag{10}$$

This form has the correct dependence on the charge q_{s-1}; it passes through the exact value $0.313(e^2/\epsilon l_0)$, for $q_0 = 1$, $m_1 = 1$, and it yields $\tilde{\epsilon}_1^{-} = 0.264$, $\tilde{\epsilon}_1^{-} = 0.0837$, for $m_1 = 3$ and $m_1 = 5$, in close agreement with the values obtained by Laughlin.[5,7]

Unfortunately, there does not exist at the present time any reliable calculation of the quasiparticle excitation energy. Therefore, *for purposes of illustration,* I have made the arbitrary approximation $\tilde{\epsilon}_s^{+} = \lambda \tilde{\epsilon}_s^{-}$, where λ is a constant independent of m_s. The resulting curve for $E(\nu)$ is plotted in Fig. 1, for the choice $\lambda = 3$, after subtraction of the "plasma approximation" $E_{\rm pl} \equiv \nu u_{\rm pl}(\nu^{-1})$, which is a smooth function of ν. We can see that there are downward pointing cusps in the energy visible at the low-order rational ν with odd denominators. The approximation also gives *upward*-pointing cusps at all rational ν with even denominators; in fact, I find small discontinuities in E, not visible on the scale of the figure, at all these even points except for $\nu = \frac{1}{2}$, where continuity is guaranteed by the particle-hole symmetry of the cohesive energy, which is respected exactly by the present approximation.[7] Clearly the upward-pointing cusps are unphysical; the system could always lower its energy by breaking up into small regions of larger and smaller density; alternatively there may be a different type of ground state with still lower energy at these values of ν. The behavior of the approximate energy curve near the low-order rationals of odd denominator should be qualitatively and semiquan-

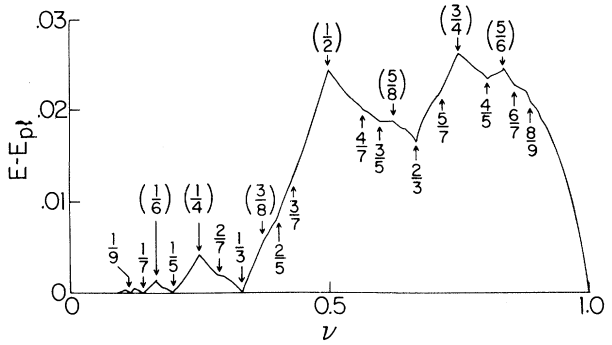

FIG. 1. Potential energy per quantum of magnetic flux, in units of $e^2/\epsilon l_0$, as a function of filling factor ν of the first Landau level, from approximate formulas (7)–(10). Smooth function $E_{\rm pl}(\nu) = \nu u_{\rm pl}(\nu^{-1})$ has been subtracted off.

titatively correct, however. More reliable estimates will be possible when p-excitation energies have been properly calculated, and when corrections are included such as the finite quasiparticle size and effects of virtual excitations of particle-hole pairs.

With the approximation described above, the energy gap $\tilde{\epsilon}_s^+ + \tilde{\epsilon}_s^-$ is equal to

$$0.313(1+\lambda)|q_s|^{5/2} m_s^{1/4}(e^2/\epsilon l_0)$$

[cf. (6) and (10)]. Except for the rather weak factor $m_s^{1/4}$, the gap is determined by the value of $|q_s|^{-1}$, which is the *denominator* of the fraction ν_s. This is in qualitative agreement with reported experimental observations on GaAs samples.[1]

Finally, we derive by induction the starting equation (1). For $s=0$, the Z_k are positions of bare electrons, and Eqs. (1) and (2) are correct, with $q_0 = m_0 = \alpha_1 = 1$. We assume that the p excitations of level $s=1$ can be formed out of *pairs* of electrons, by a generalization of Eq. (23) of Ref. 3. A system containing N_1 pairs of electrons, together with $N_0 - 2N_1$ unpaired electrons, is then described by choosing the polynomial in (1) to have the (schematic) form

$$P[Z_k] = \mathcal{S}\bar{P}[z_i] \prod_{i<j}(z_i - z_j)^{8p_1-4} \prod_{i,\gamma}(z_i - \tilde{Z}_\gamma)^{4p_1-1} \prod_{\gamma<\delta}(\tilde{Z}_\gamma - \tilde{Z}_\delta)^{2p_1}, \tag{11}$$

where z_i are the positions of the centers of gravity of the bound pairs, $\tilde{Z}_\gamma$ are the positions of the *unpaired* electrons, $\bar{P}$ is a symmetric polynomial, and $\mathcal{S}$ is an operator which symmetrizes with respect to the positions of all N_0 electrons. I have assumed that the separation between two members of a pair is small, and have dropped the variables describing these separations. To calculate the probability distribution of the pairs, we ignore the symmetrizer $\mathcal{S}$, and take the trace of $|\Psi[Z_k]|^2$ over the unpaired electron positions $\tilde{Z}_\gamma$. The result can be expressed in the form $|\bar{\Psi}[z_i]|^2\Phi[z_i]$, where $\bar{\Psi}$ has again the form of (1) and (2), with P replaced by $\bar{P}$, and with $m_1 = 2p_1 - 1/m_0$, $\alpha = 1$, and $q_1 = q_0/m_1$, while the remaining factor Φ is the partition function of a classical one-component plasma with sources of strength $2 - m_1^{-1}$, located at the positions z_i. Now Φ will be independent of the positions z_i, provided that the sources are sufficiently separated so that their screening clouds do not overlap. Thus it is consistent to interpret $\bar{\Psi}$ as a pseudo wave function for the positions of the pairs. Higher levels may be obtained iteratively.

Derivation of the pseudo wave function for hole excitations is more complicated because of the necessity to use an integral representation, such as Eq. (25) of Ref. 3.[7]

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