Solitons in Superfluid ^{3}He-A: Bound States on Domain Walls

T. L. Ho,[a] J. R. Fulco, J. R. Schrieffer, and F. Wilczek
Institute for Theoretical Physics and Department of Physics, University of California, Santa Barbara, California 93106
(Received 16 January 1984)

The effects of solitons on the spectrum of fermion excitations in superfluid ^{3}He-A are investigated. It is found that there is a two-dimensional manifold of bound states with energies within the gap of the bulk superfluid. The bound-state spectrum lacks inversion symmetry parallel to the wall.

PACS numbers: 67.50.Fi

There is considerable interest in broken-symmetry fermion systems exhibiting topological soliton excitations. It has been shown that the phenomenon of topologically generated fermion bound states is quite general. For example, in relativistic quantum field theories[1] and in quasi-one-dimensional conductors[2] there are zero-energy fermion states bound to the soliton, which thereby acquires fractional or even irrational fermion number depending on the ground-state degeneracy of the system.[3,4] Fermion bound states also exist in vortices in type-II superconductors,[5] where the vortex singularity provides the trapping potential. On the other hand, in the case of superfluid ^{3}He, where there are nonsingular topological solitons, it is not obvious whether all (or any) of them bind quasiparticles.

In this Letter, we study the quasiparticle bound states associated with planar solitons in ^{3}He-A. We find that the spectrum of bound states has many branches which are confined to a two-dimensional manifold in momentum space. The spectra have some features similar to those of charge-density-wave (CDW) systems,[2] in that for certain values of the momentum there is a zero-energy bound state. We also find that there is a lack of inversion symmetry of the bound-state spectrum in momentum space.

^{3}He-A is a p-wave BCS superfluid consisting of Cooper pairs with unit orbital angular momentum along an axis $\hat{l}$. These are spin triplets (parallel spin pairing) which can be regarded as a state having spin perpendicular to an axis $\hat{d}$.[6] The excitation spectrum has an anisotropic gap $\Delta(\vec{p})=[1-(\hat{l}\cdot\hat{p})^2]^{1/2}\Delta$ at the Fermi level $|\vec{p}|=p_F$. Unlike the CDW and relativistic theories mentioned above, the condensate field of ^{3}He-A is a tensor (in spin and orbit space) rather than a scalar.[6] As a result, there are many types of solitons in ^{3}He-A.

In the following, we consider a simplified version of the "composite" soliton.[7] It is a planar structure where $\hat{d}$ is uniform and $\hat{l}$ rotates through an angle $\pi-2\theta_0$ as one moves from $-\infty$ to $+\infty$. The width of the soliton is fixed by the dipolar force and is of order 1 μm.

To understand qualitatively why bound states exist at all in composite solitons, consider the case in which $\theta_0=0$ and $\hat{l}$ rotates continuously from $-\hat{y}$ to $\hat{x}$ to $\hat{y}$ in the x-y plane as $\hat{z}$ varies from $-\infty$ to 0 to $+\infty$, with $\Delta_-\rightarrow\Delta_0\rightarrow\Delta_+$. Near $z=0$, where $\hat{l}=\hat{x}$, the "local" excitations traveling nearly along the x direction with $|\vec{p}|\simeq p_F$ will have energies $E(p)<|\Delta_0(p)|<|\Delta_\pm|$. Since these states lie in the forbidden region of the spectra at $\pm\infty$, they form bound states around $z=0$.

Consider the time Fourier transform of the Gor'kov-Nambu equations for p-wave superfluids:

$$\int d^3r'[\omega\delta(\vec{r}-\vec{r}\,')-H(\vec{r},\vec{r}\,')]\begin{pmatrix} G(r,r',\omega)\\ \bar{F}(r,r',\omega)\end{pmatrix}=\delta(\vec{r}-\vec{r}\,')\begin{pmatrix}1\\0\end{pmatrix}, \tag{1}$$

$$H(\vec{r},\vec{r}\,')=\begin{pmatrix} h_0 & \Delta(r,r')\\ \Delta^+(r,r') & -h_0\end{pmatrix}, \tag{2}$$

where G, $\bar{F}$, and Δ are spin matrices; $G_{\mu\nu}(\omega)$ and $\bar{F}_{\mu\nu}(\omega)$ are time Fourier transforms of the Green's functions

$$G_{\mu\nu}(\vec{r},\vec{r}\,',t)=-i\langle T\psi_\mu(\vec{r},t)\psi_\nu^\dagger(\vec{r}\,',0)\rangle,$$

$$F_{\mu\nu}(\vec{r},\vec{r}\,',t)=-i\langle T\psi_\mu^\dagger(\vec{r},t)\psi_\nu^\dagger(\vec{r}\,',0)\rangle;$$

$h_0=(-\nabla^2/2m)-\mu$; and μ is the chemical potential. The gap function is given in terms of the two-body potential as

$$\Delta_{\mu\nu}(\vec{r},\vec{r}\,')=-V(\vec{r}-\vec{r}\,')\langle\psi_\mu(\vec{r})\psi_\nu(\vec{r}\,')\rangle.$$

In the A phase, when the spin variable is a constant (say, $\hat{d}=\hat{y}$ in the conventional notation),[6] we have $\Delta_{\mu\nu}(\vec{r},\vec{r}\,')=\Delta(\vec{r},\vec{r}\,')\delta_{\mu\nu}$ $[\Delta(\vec{r},\vec{r}\,')=-\Delta(\vec{r}\,',$

$\vec{r})]$. The quantities G, $\bar{F}$, and Δ in (1) and (2) then become scalars describing the parallel spin pairing. The Green's functions G and $\bar{F}$ can be constructed from the eigenfunctions of H.

For a spatially varying Δ, the integral equation $H\chi = E\chi$ is difficult to solve. Considerable simplification can be made if we focus on the excitations near the Fermi level. The spatial transform of the gap function can therefore be approximated by[8]

$$\int d^3r\Delta(\vec{R}+\vec{r}/2, \vec{R}-\vec{r}/2)e^{-i\vec{p}\cdot\vec{r}} = i(\Delta/p_F)\vec{p}\cdot\hat{\phi}(\vec{R}), \quad (3)$$

where $\hat{\phi}(\vec{R}) = \hat{w}_1(\vec{R}) + i\hat{w}_2(\vec{R})$ $(\hat{w}_1\cdot\hat{w}_2 = 0,\ \hat{w}_1^2 = \hat{w}_2^2 = 1)$ is the local order parameter, and $\hat{l} = \hat{w}_1 \times \hat{w}_2$ is the "angular momentum" texture. With (3), the integral equation $H\chi = E\chi$ can be written as

$$(E_n - h_0)u_n = (\Delta/p_F)(\tfrac{1}{2}\nabla\cdot\hat{\phi} + \hat{\phi}\cdot\nabla)v_n, \quad (4)$$

$$(E_n + h_0)v_n = -(\Delta/p_F)(\tfrac{1}{2}\nabla\cdot\hat{\phi}^* + \hat{\phi}^*\cdot\nabla)u_n, \quad (5)$$

where $\chi = (u_n, v_n)$. We search for solutions of the form

$$(u_n, v_n) = [u_n^0(z), v_n^0(z)]e^{i\vec{p}\cdot\vec{r}}.$$

For bound states, u_n^0 and v_n^0 decay as $\exp(-\kappa_\pm|z|)$ as $|z| \to \pm\infty$, with $\kappa_\pm^{-1}$ of order of the coherence length $\xi_0 = p_F/\pi m\Delta$, which is large compared to p_F^{-1}. Thus, for bound states or scattering states near the Fermi surface we neglect terms of order $\xi_0^{-1}p_F \approx 10^{-3}$ and Eqs. (4) and (5) become

$$[E_n + i(p_z/m)\partial_z - \epsilon_p]u_n^0 = (\Delta/p_F)(\tfrac{1}{2}\nabla\cdot\hat{\phi} + i\hat{\phi}\cdot\vec{p})v_n^0, \quad (6)$$

$$[E_n - i(p_z/m)\partial_z + \epsilon_p]v_n^0 = -(\Delta/p_F)(\tfrac{1}{2}\Delta\cdot\hat{\phi}^* + i\hat{\phi}^*\cdot\vec{p})u_n^0, \quad (7)$$

where $\epsilon_p = p^2/2m - \mu$.

We first consider order parameters in which $\hat{w}_2$ is fixed along $\hat{z}$ and $\hat{w}_1$ rotates in the x-y plane (case a),

$$\hat{\phi}(z) = w_1(z) + iw_2(z) = \hat{x}\cos\theta(z) + \hat{y}\sin\theta(z) + i\hat{z}, \quad (8)$$

$$\nabla\cdot\phi = 0,$$

with the boundary conditions $\theta(-\infty) = \pi - \theta_0$ and $\theta(\infty) = \theta_0$, where $0 \leqslant \theta_0 \leqslant \frac{1}{2}\pi$. We note that, at $z \to \pm\infty$, the bound state solutions of Eqs. (6) and (7) are of the form $\exp(-\kappa_\pm|z|)(u,v)_\pm$, where κ_+ and $\kappa_- > 0$. The fact that $E_n \equiv E(\vec{p})$ is real implies that $\epsilon_p = 0$, i.e., $p^2 = p_F^2$.

Equations (6) and (7) have solutions with the formal symmetry

$$E(-\vec{p}) = -E(\vec{p}), \quad (u^0, v^0)_{-\vec{p}} = (v^{0*}, u^{0*})_{\vec{p}}. \quad (9)$$

In addition, for $\hat{\phi}$ given by Eq. (8), there is the added formal symmetry

$$E(-p_x, -p_y, p_z) = -E(p_x, p_y, p_z), \quad (u^0, v^0)_{-p_x, -p_y, p_z} = (-v^0, u^0)_{p_x, p_y, p_z}. \quad (10)$$

It is essential to realize that the physical excitation spectrum is given by the *positive* eigenvalues $E(\vec{p}) > 0$, while the *negative* eigenvalues $E(-\vec{p}) < 0$ correspond to the energy lowering when an excitation of momentum $+\vec{p}$ is destroyed.

Because of Eqs. (9) and (10) we consider only the case $\hat{p}\cdot\hat{z} \geqslant 0$. Defining $\zeta = z/\xi_0$, it follows that $(p_z/m)\partial_z = \hat{p}\cdot\hat{z}\Delta\partial_\zeta$, and Eqs. (6) and (7) become

$$(\tilde{E}+1)a = [\partial_\zeta + F(\zeta)]b, \quad (\tilde{E}-1)b = [-\partial_\zeta + F(\zeta)]a, \quad (11)$$

where

$$E(\vec{p}) = \hat{p}\cdot\hat{z}\Delta\tilde{E}, \quad F(\zeta) = (\hat{p}\cdot\hat{w}_1)/(\hat{p}\cdot\hat{z}), \quad (12)$$

with $a = u^0 + v^0$, $b = -i(u^0 - v^0)$. From Eqs. (11) one obtains

$$(\tilde{E}^2 - 1)\begin{pmatrix} a \\ b \end{pmatrix} = \left[-\partial_\zeta^2 + \begin{pmatrix} V_a \\ V_b \end{pmatrix}\right]\begin{pmatrix} a \\ b \end{pmatrix}, \quad (13)$$

where $V_{a(b)} = F^2 + (-)\,\partial_\zeta F$.

For a sharp wall one finds the remarkable result that a physical excitation exists only for $p_x > 0$, with $p^2 = p_F^2$, and has energy

$$E(\vec{p}) = |\hat{p}\cdot\hat{z}|\Delta. \quad (14)$$

This result is a consequence of V_b being an attractive (repulsive) delta function for p_x positive (negative). When the width of the wall increases, this "zeroth branch" solution persists but a finite number of higher-energy branches appear for both positive and negative p_x (see Fig. 1).

The components of the wave function for the zeroth branch are

$$a = 0, \quad b(\zeta) = b(0)\exp[-\int_0^\zeta ds\, F(s)]. \quad (15)$$

For this solution to exist, the following conditions must be satisfied:

$$\hat{p}\cdot\hat{w}_1(+\infty) > 0; \quad \hat{p}\cdot\hat{w}_1(-\infty) < 0. \quad (16)$$

Note that as long as $F(\zeta) \gtrless 0$ for $\zeta \to \pm\infty$ the

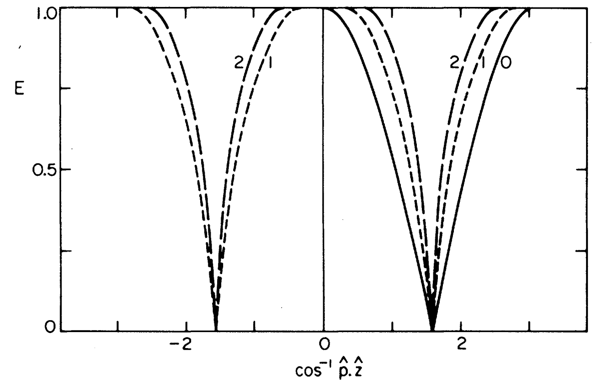

FIG. 1. The spectrum of bound states $E(\vec{p})$ for case a: $\hat{\phi}(z)=\hat{x}\cos\theta(z)+\hat{y}\sin\theta(z)+i\hat{z}$ and $n=0,1,2$.

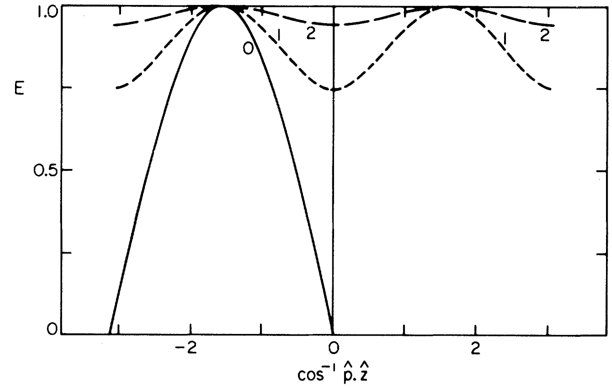

FIG. 2. The spectrum of bound states $E(\vec{p})$ for case b: $\hat{\phi}(z)=-\hat{z}\cos\theta(z)+\hat{y}\sin\theta(z)+i\hat{x}$ and $n=0,1,2$.

zeroth branch is independent of the shape of the soliton.

Next we consider order parameters in which $\hat{w}_2$ is fixed along $\hat{x}$ (case b),

$$\hat{\phi}(z)=-\hat{z}\cos\theta(z)+\hat{y}\sin\theta(z)+i\hat{x},\qquad \nabla\cdot\phi\neq 0. \tag{17}$$

Since Eq. (17) is obtained from Eq. (8) by rotating $\hat{\phi}(z)$ by $\pi/2$ about the $\hat{y}$ axis, it is easy to see that Eqs. (11) and (12) still apply and Eqs. (13) are replaced by

$$(\tilde{E}+\gamma)a=(\partial_\zeta+F+i\delta\,\partial_\zeta\cos\theta)b,\qquad (\tilde{E}-\gamma)b=(-\partial_\zeta+F+i\delta\,\partial_\zeta\cos\theta)a, \tag{18}$$

where $\delta=(\Delta/4E_F)(\hat{p}\cdot\hat{z})^{-1}$, $\gamma=(\hat{p}\cdot\hat{x})/(\hat{p}\cdot\hat{z})$, and $F(\zeta)=(\hat{p}\cdot\hat{w}_1)/(\hat{p}\cdot\hat{z})$. All other quantities are defined as before. Again $|\vec{p}|^2=p_F^2$ and we take $\hat{p}\cdot\hat{z}\geqslant 0$.

The zeroth-branch solution is

$$b=0,\quad a(\zeta)=e^{i\delta\cos\theta}\exp\left[-\int_0^\zeta ds\,F(s)\right] \tag{19}$$

and the excitation spectrum is (see Fig. 2)

$$E_{n=0}(\vec{p})=-\hat{p}\cdot\hat{x}\Delta\quad\text{for } \hat{p}\cdot\hat{x}<0. \tag{20}$$

We next calculate the change $\Delta\rho$ of the bare fermion density integrated over p_z for fixed $p_x>0$, due to the existence of the soliton centered at $z=0$. To this end we introduce rigid-wall boundary conditions at $z=\pm a$ and periodic boundary conditions on x and y. For example, for case a and a sharp wall

$$\Delta\rho_{p\perp}=\sum_{p_z'}|v(\vec{p}\,')|^2-\sum_{p_z}|v(\vec{p})|^2, \tag{21}$$

where

$$|v(\vec{p})|^2=\tfrac{1}{2}[1-\epsilon(\vec{p})/E(\vec{p})] \tag{22}$$

and

$$p_{z_n}=(2n+1)\pi/2a,\qquad p_{z_n}'=(2n+1)\pi/2a+\alpha(\vec{p}\,')/a, \tag{23}$$

where

$$\alpha(\vec{p}\,')=-\tan^{-1}[(\hat{p}\cdot\hat{w}_1^+)/(\hat{p}\cdot\hat{z})].$$

In the limit $p_Fa>>1$, the change in the bare fermion density arising from the scattering states for $p_x>0$ is

$$\Delta\rho_{p\perp}=\frac{1}{\pi}\int_0^\infty dp_z\frac{d(\alpha|v(\vec{p})|^2)}{dp_z}=\frac{1}{2}, \tag{24}$$

while the corresponding quantity for $p_x<0$ is $-\frac{1}{2}$.

Therefore the integrated bare fermion density arising from the scattering states is unchanged. However, since $|v_p^0|^2=\frac{1}{2}$ for the bound states there would be an accumulation of half a bare fermion state, in the ground state, for each $p_\perp$.

In conclusion, we find that a planar soliton in ^{3}He-A exhibits a two-dimensional manifold of bound states in the quasiparticle spectrum. This spectrum has two types of branches: the $n=0$ branch and $n\geqslant 1$ branches. The former has the surprising property that its spectrum lacks inversion symmetry for $p_x\to -p_x$, where x is parallel to the domain wall. As a consequence, at $T>0$ these quasiparticle bound states will be occupied and quasiparticle current will flow parallel to the wall in the direction $\hat{l}_-\times\hat{z}$, where $\hat{l}_-$ is the angular momentum in the region $z<0$ far from the wall. In the typical experimental situation in which many solitons are produced, there are broad antisolitons separating the solitons. Our solutions show that, at the antisolitons, an anomalous quasiparticle current will flow in the reverse direction. These currents are connected at the container wall, so that total current is conserved.

This work was supported by National Science Foundation Grants No. PHY80-18938, No. DMR82-16285, and No. PHY77-27084 (supplemented by funds from the National Aeronautics and Space Administration), and the Physics Department of the Ohio State University. One of us (T.L.H.) would like to thank the Physics Department and the Institute for Theoretical Physics at the University of California at Santa Barbara for their hospitality during the summer of 1983. One of us (J.R.F.) would like to acknowledge the hospitality of the Centre de Physique Théorique–Ecole Polytechnique and the University de Paris-Sud, Orsay, France where part of this work was done.

---

[a]Permanent address: Department of Physics, Ohio State University, Columbus, Ohio 43210.

[1]R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976).

[2]W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. **42**, 1698 (1979), and Phys. Rev. B **22**, 2099 (1980); R. Jackiw and J. R. Schrieffer, Nucl. Phys. **B190** [FS3], 253 (1981).

[3]W. P. Su and J. R. Schrieffer, Phys. Rev. Lett. **46**, 738 (1981).

[4]J. Goldstone and F. Wilczek, Phys. Rev. Lett. **47**, 986 (1981).

[5]P. G. de Gennes, *Superconductivity of Metals and Alloys* (Benjamin, New York, 1966), p. 153.

[6]See, for example, A. J. Leggett, Rev. Mod. Phys. **47**, 331 (1975).

[7]K. Maki and P. Kumar, Phys. Rev. B **14**, 118 (1976), and Phys. Rev. Lett. **38**, 557 (1977); C. M. Gould and D. M. Lee, Phys. Rev. Lett. **37**, 1223 (1976).

[8]The Fourier transform of (3) should be $f(\vec{R}, |\vec{p}|) i\Delta\hat{p}\cdot\hat{\phi}(\vec{R})$, where $f \approx 1$ near p_F and vanishes beyond a cutoff ω_c about the Fermi level. Our approximation reproduces the bulk spectrum near $|\vec{p}| = p_F$. It amounts to suppressing the spatial variation of the magnitude of the gap and requiring $\omega_c/\Delta >> 1$. The approximation is reasonable since, as we shall see, the bound states only involve states with $|\vec{p}| = p_F$.