Topological Mapping Properties of Collisionless Potential and Vortex Motion

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(Received 24 May 1983)

The topological properties of patterns which arise in free motion of matter under random but smooth initial velocity distributions are considered. Potential and solenoidal motions are discussed. The cosmological role of this phenomenon is also briefly discussed.

PACS numbers: 47.10.+g, 47.30.+s, 98.80.–k

Let us consider the free motion of a continuous pressureless medium with a random but smooth initial velocity field and constant initial density $\rho(\vec{x},t=0)\equiv 1$. The problem is interesting being similar to the cosmological motion leading to the formation of structure in the universe, at least in some variants of the theory.[1–3]

In Lagrangian coordinates and in the simplified force-free case this motion is described by the equation

$$\vec{x}=\vec{\xi}+t\vec{v}(\vec{\xi}), \tag{1}$$

here $\vec{x}$ are Eulerian and $\vec{\xi}$ Lagrangian coordinates, t is the time, and $\vec{v}(\vec{\xi})$ is the velocity of every particle specified at $t=0$ and constant for every particle thereafter. The motion perturbs the uniformity of the density distribution. With a smooth random velocity field, at small t density perturbations are small and are given by a smooth function. The general features of the strongly nonlinear situation are, however, nontrivial. The growth of negative perturbations is limited, $\delta\rho/\rho > -1$, because mass density is never negative. On the other hand positive perturbations can grow without limit and it turns out that infinite density is achieved in a finite time as a result of caustic formation (as in catastrophe theory). The density distribution in Eulerian space is also highly asymmetric: The dense regions (separated from rarified ones by caustics) occupy a volume which is several times less than that occupied by rarified regions. This is evident from mass conservation. The nontrivial point that we wish to stress concerns the topology of the dense regions at the nonlinear stage: They form thin walls which surround rarified regions—the rarified domain consists of large separated regions. There are infinite jumps of density on the frontiers between dense and rarified domains. This structure is born out of a smooth velocity distribution without singularities. We believe that these features of the simplified problem are reflected in the pattern observed by astronomers: the large black regions ("voids") without galaxies and the strings of galaxies.

The two-dimensional case can be modeled by the refraction or reflection of a parallel light beam by a smooth random surface. Thin bright regions can be observed in a sunny day at the bottom of a swimming pool (Fig. 1). The occurrence of caustics in light propagation is well known—their connection with catastrophe theory has already been mentioned.[4,5] However, their topological properties—in particular, the formation of a network—have not been studied theoretically.

The peculiar topological properties of the density distribution occur only as an intermediate asymptote. They are absent in the linear regime and they vanish after multiple crossing of light rays or trajectories of collisionless particles. If we are dealing with gravitating particles (see below) the dense regions remain thin—gravity produces an effective sticking.[6,7] However, these regions become unstable against disruption of their surfaces. Again the cellular or network structure exists as an intermediate asymptotic. The ultimate state consists of isolated but continually merging dense regions (clumps).[6,8–10] The duration of the peculiar cellular or network structure depends on the power spectrum of the initial velocity perturbations. The search for specific structural and topological properties of a given density distribution is rather difficult, especially when working with discrete objects—galaxies—instead of a continuous medium. The methods of percolation theory[11–13] are well suited to this case.

At first glance the motion of a cosmological medium under gravitation and the Hubble expansion is quite different from that of a noninteracting medium. However, it has been shown[14] that cosmological motion has much in common with simple inertial motion. From the decoupling of matter and radiation until the formation of large-

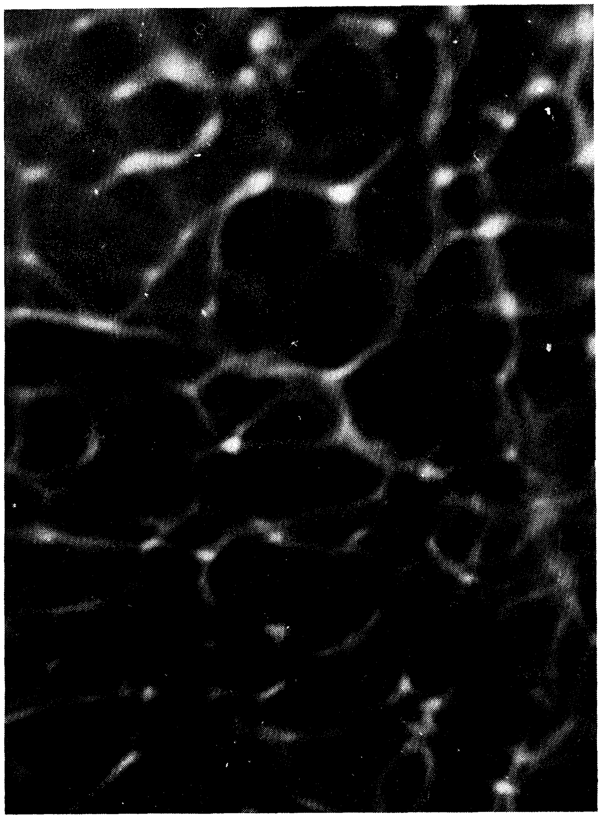

FIG. 1. The photograph of a screen illuminated by light reflected by the wavy surface of a water pool. The surface is lighted by a parallel train. The bright regions in the picture correspond to the density regions in the case of collisionless particles.

scale structure, the motion of a gas of slow massive neutrinos or of neutral atoms can be described by a simple approximate solution

$$\vec{r} = a(\tau)\vec{\xi} + b(\tau)\mathrm{grad}\phi(\vec{\xi}), \tag{2}$$

here $\vec{r}$ and $\vec{\xi}$ are Eulerian and Lagrangian coordinates, $a(\tau)$ is the scale factor, and τ is the time. The time dependence of $a(\tau)$ describes the Hubble expansion of the universe. The second term on the right-hand side of (2) describes the perturbations. The formula assumes that only the growing mode $b(\tau) \propto \tau^{4/3}$ is present. The perturbations are growing because of gravitational instability and therefore the velocity field must remain irrotational. One can thus describe it by a potential $\phi(\vec{\xi})$ which can be derived from initial perturbations. Making simple substitutions

$$\vec{r} = a(\tau)\cdot\vec{x}, \quad b(\tau)/a(\tau) = t, \quad \mathrm{grad}\phi(\vec{\xi}) = \vec{v}(\vec{\xi}),$$

one can easily obtain (1) from (2).

At the stage when structure forms the behaviors of neutrinos and normal matter (atoms) become different. Neutrinos are collisionless and hence regions of multistream flow must occur. Atoms constitute an ordinary gas which experiences shock compression. However, as it was mentioned above, gravity acts like sticking; therefore we expect that the difference between these two cases is small. The simple formula (2) is equally inapplicable in both cases in the compressed regions.

In this Letter we consider the most general potential flow and discuss also solenoidal velocity fields. There are eight cases of the problem which originate from three alternatives of choice: (1) potential (P) or vortex (V) velocity field of (2) collisionless (C) or sticking (S) medium in (3) two-dimensional or three-dimensional (3D) space. Earlier, S cases were briefly discussed by one of the present authors.[15]

Let us firstly consider all PC cases in 2D and 3D. In these cases

$$\begin{aligned} v_i &= \partial\phi/\partial\xi_i \\ \rho &= \mathscr{J}^{-1} = |\delta_{ik} + t\,\partial^2\phi/\partial\xi_i\,\partial\xi_k|^{-1}. \end{aligned} \tag{3}$$

While t is small (summing over i),

$$\rho \approx 1 - t\,\partial^2\phi/\partial\xi_i\,\partial\xi_i. \tag{4}$$

We consider random but smooth, statistically homogeneous functions ϕ with a power spectrum cut off at both short and long wavelengths and with random and independent Fourier phases; the distribution function of ϕ is then Gaussian.

Let some level ρ_0 divide space onto two domains: a dense one where $\rho > \rho_0$ and a rarified one where $\rho < \rho_0$. What can one say about the topology of these domains? The answer comes from percolation theory. Usually in 2D the domain which contains more than 50% of the total area contains the infinite cluster (or in other words there is percolation of this phase) which divides the other domain (occupying less than 50% of the surface) into separate regions (there is no percolation of this phase). In 3D there is also the possibility that both domains contain infinite clusters. This takes place if one of the domains occupies more than $\approx 16\%$ but less than 84% of the total volume. In other cases the smaller domain consists of separated regions.[10] These two critical values coincide quite accurately with levels $\rho_0 = \bar{\rho} \pm \sigma_\rho$.

At the linear stage of perturbation growth, $\langle(\rho - \bar{\rho})^2\rangle << \bar{\rho}^2$, the positive $(\rho - \bar{\rho} > 0)$ and negative $(\rho - \bar{\rho} < 0)$ domains are statistically symmetric and each occupies 50% of the surface or volume. In 2D both phases are marginally percolating, in 3D they are strongly percolating.

The situation becomes quite different at the nonlinear stage. Instead of (4) one gets the following equation for density (in 2D):

$$\rho = \mathscr{I}^{-1} = (1 + tI_1 + t^2 I_2)^{-1} = (1 - t\alpha)^{-1}(1 - t\beta)^{-1}, \tag{5}$$

where

$$I_1 = \partial v_1/\partial\xi_1 + \partial v_2/\partial\xi_2,$$

$$I_2 = (\partial v_1/\partial\xi_1)\partial v_2/\partial\xi_2 - (\partial v_1/\partial\xi_2)\partial v_2/\partial\xi_1,$$

which in P case coincide with the invariants of $\partial^2\phi/\partial\xi_i\partial\xi_k$; α and β are the eigenvalues of $\partial^2\phi/\partial\xi_i\partial\xi_k$.

A mapping from ξ space to x space, where the structure is observed, is produced by Eq. (1). It is continuous and therefore conserves an important topological property: Closed curves or surfaces in Lagrangian space remain closed in Eulerian space in spite of their strong deformation (even if additional intersections do arise).

Let us consider first the Lagrangian space. About 79% of the total area in 2D and 92% of the total volume in 3D is occupied by matter which possesses at least one positive eigenvalue.[17] In spite of the fact that the statistics of the eigenvalues are not Gaussian it is thus quite reasonable to suppose that the domain of at least one positive eigenvalue contains the infinite cluster, and so we can say that there is percolation along the regions of positive eigenvalue (Fig. 2). After mapping into Eulerian space these regions contract and therefore occupy less area (in 2D, Fig. 3) or volume (in 3D), but they remain connected to each other and we can speak about the conservation of percolation (topological) properties. Just this very reason explains why at the nonlinear stage the thin regions of compressed matter separate regions of low density in 2D, in spite of the former occupying in 2D space much less than 50% and the latter more than 50% of the total area (see also the optical example in Fig. 1).

In 3D the equation $\mathscr{I} = 0$ becomes third order which has for the P case, three real roots for t: in 8% of all cases three positive roots; in 42%, two positive and one negative; also in 42%, one positive and two negative; and in 8%, three negative roots.[17] This means that about 92% of the matter will experience contraction in caustics which is larger than the 84% that is necessary for the isolation of rarified regions in the Gaussian case.

Let us now discuss the V case, i.e., a vortex velocity field. In 2D one can produce this field from a flux function ϕ,

$$v_i = \partial\phi/\partial\xi_2; \quad v_2 = -\partial\phi/\partial\xi_1. \tag{6}$$

In this case we get $I_1 \equiv 0$ and

$$\mathscr{I} = 1 + I_2 t^2. \tag{7}$$

Remembering that in the potential case the signs of the eigenvalues α and β depend on the sign of I_2, we immediately find that regions of negative I_2 coincide with those where α and β have opposite signs (Fig. 2). This means that about 58% of matter will pass through infinite density. This value shows that it is very probable that the domain of negative I_2 contains an infinite cluster and percolates.

In the 3D V case the equation $\mathscr{I} = 0$ has one or three real roots for t with probabilities p_1 and p_3, $p_1 + p_3 = 1$. It is clear from symmetry that the probability of a positive root equals $p_+ = 0.5p_1 + AP_3$,

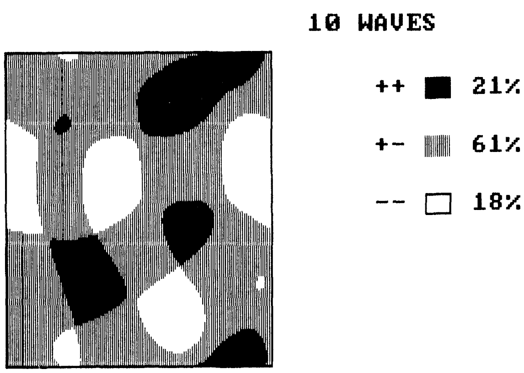

FIg. 2. The example of a random perturbation field. Black areas are regions where both eigenvalues (α and β) are positive; grey, α is positive but β is negative; and white, α and β are negative. Theoretical values for the areas are 21% (+ +), 58% (+ −), and 21% (− −). In the realization these numbers are a little different.

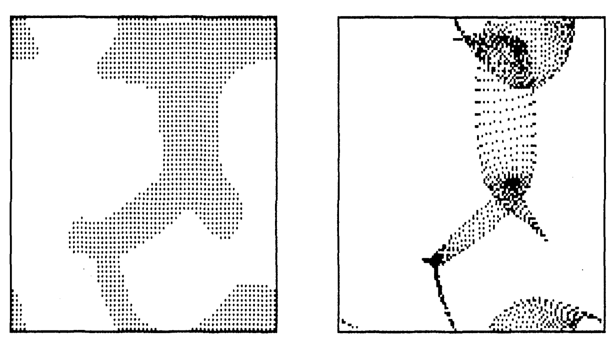

FIG. 3. (a) The region of $\alpha > \alpha_0$ in Lagrangian space, occupying about one-third of the total area. (b) The mapping of this region into Eulerian space at $t = 1/\alpha_0$; in Eulerian space it occupies only about one-fifth of the total area.

with $A > 0.5$. Thus in the 3D V case more than 50% of matter experiences contraction in caustics (i.e., reaches infinite density at some time). This is enough for formation of a connected network. However, it is probably not enough for separation of rarified regions. Thus it seems that both phases (dense and rarified) contain infinite clusters (i.e., percolate) which of course is impossible in 2D. As was mentioned for C cases, the network structure is an example of intermediate asymptotics. In the limit $t \rightarrow \infty$ at each point $\vec{x}$ there are more and more particles from quite different regions of $\vec{\xi}$. A Gaussian distribution of initial velocity field in space becomes a Maxwellian distribution of velocities at every point in this limit. Density fluctuations are close to those in a system at thermodynamic equilibrium. However, a finite limit arises only if continuous medium is replaced by a set of point particles with finite mass as the temperature of the system is proportional to the mean kinetic energy of a single particle. In a continuous medium reasonably defined density fluctuations in an arbitrarily small but fixed volume tend to 0 as $t \rightarrow \infty$.

The fact that the structures we discuss are temporary does not make the problem uninteresting. It seems that we are lucky to live at the time when the large-scale structure of the universe is similar to one of the cases discussed, namely, 3D-P-S. Destruction of the structure will need more time than the present age of the universe. Phenomena of such importance and duration undoubtedly deserve consideration.

The authors are grateful to V. I. Arnold and S. D. M. White for discussions, A. Szalay for discussions and making the computer illustrations, and A. Mamaev for making the optical illustration.

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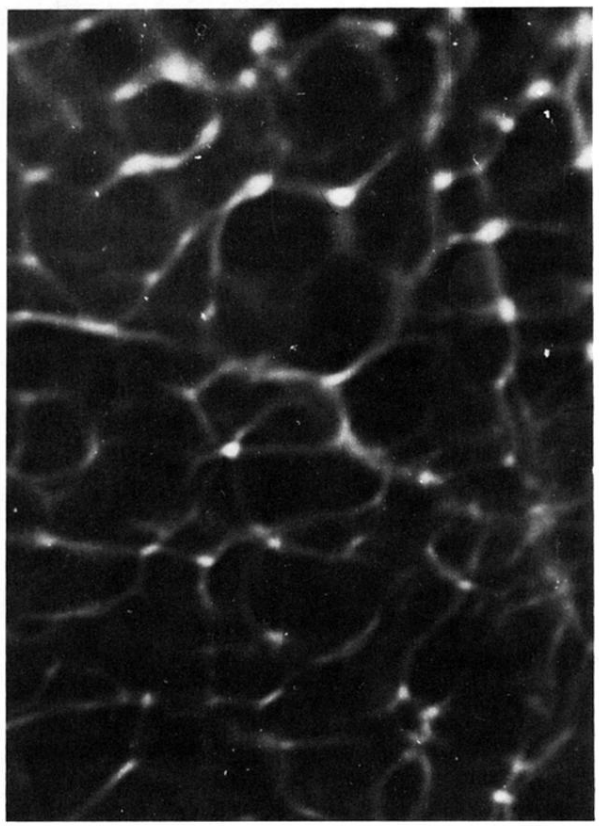

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